WP3.2: Statistical analysis of test particle transport

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- Since the past two decades, an outstanding effort has been invested to understand non-diffusive, non-Gaussian scaling of the radial dispersion of passive particles within models of three-dimensional, resistive, pressure-gradient-driven plasma turbulence in both cylindrical [44: del-Castillo-Negrete et al. PP 2004] and toroidal [45: Garcia and Carreras PP 2006] geometris. Here, we propose to take the next step, and to extend the statistical analysis of the radial displacements of passive particles to the different EP transport models used within this project.

- The bottleneck problem is whether the probability density function of the radial displacements of passive tracers reveals an algebraically decaying tail and whether the moments of the tracer displacements exhibit super-diffusive scaling.

- Tracers dynamics will be studied with different techniques. In particular the Lagrangian Coherent structures theory [46: Falessi et al. JPP 2015) will be applied to study the connections between superdiffusive/convective transport and the formation of structures in the phase space.

WP3.2: Statistical analysis of test particle transport

- Lately, there had been theory-based premises that the transport could be just convective, with finite second moments [43: Carlevaro et al. Entropy 2016]. The finiteness of the second moments implies, in its turn, that the Lévy flights do not step in into the dynamics, so the relaxation process is local, i.e., fluxes at a point are defined by gradients at the same point. Theoretically, this is a very strong statement and needs to be verified in a simulation. This will be performed using the hierarchy of different models used in the different WPs of this project.

- Should the results of this verification reveal the presence of far-reaching power-law tails, then the oft-used assumption of locality will result invalidated. In this case, more sophisticated models should be introduced, with room for non-local transport with Lévy flights (as e.g. in ref. [47: Milovanov and Rasmussen PRE 2018). At this point (if we in fact arrive at this point), we would like to know the exact exponent of the power-law, and possibly also the theoretical arguments justifying this law. We expect these and related investigations to take effect during the third year of the project.

Superdiffusion and Viscoelastic Vortex Flows in a Two-Dimensional Complex Plasma

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Viscoelastic vortical fluid motion in a strongly coupled particle system has been observed experimentally. Optical tracking of particle motion in a complex plasma monolayer reveals high grain mobility and large scale vortex flows coexistent with partial preservation of the global hexagonal lattice structure. The transport of particles is superdiffusive and ascribed to Lévy statistics on short time scales and to memory effects on the longer scales influenced by cooperative motion. At these longer time scales, the transport is governed by vortex flows covering a wide spectrum of temporal and spatial scales.

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"Complex plasmas" are obtained when microparticles are added to a weakly ionized gas. Under certain conditions, these charged microparticles can arrange themselves in a regular lattice structure governed by electrostatic interaction [1,2]. In certain cases, phase transition from this crystalline to a disordered state passes through the "flow and floe" state characterized by islands of crystalline order surrounded by streams of particles [3]. Systems studied in other experiments [4–6] exhibit ordered hexagonal structure of the entire system, and yet the grains are relatively mobile. References [4,5] reported subdiffusive transport on short time scales, due to caging of the particles, and slightly superdiffusive transport on intermediate time scales, due to the emergence and relaxation of crystal defects. In those experiments, these motions tend towards normal diffusion on even longer time scales, while superdiffusive particle transport on all time scales up to the limit given by the finite system size was reported in Ref. [6]. On the other hand, no large scale fluidlike motion has been observed in these systems. Laminar and turbulent fluid flows have been observed in several complex-plasmas experiments (e.g., [7]) but only in systems in the liquid or gaseous states.

In this Letter, we report observations of a partly ordered state in a complex plasma monolayer that allows hydrodynamic vortical flows. This state is observed in a narrow range of neutral gas pressures; the monolayer freezes at higher as well as lower pressure. The dynamics displays elastic deformation on short temporal and spatial scales but looks more like a viscous flow on larger scales; the essential characteristics of a viscoelastic flow. The basic idea of viscoelasticity can be understood in terms of the Maxwell model, where the strain γ consists of two components $\gamma = \gamma_e + \gamma_v$. The elastic component γ_e responds to the stress σ through Hooke's law $\sigma = G_0\gamma_e$ and the viscous component γ_v through the friction relation $\sigma = \eta\dot{\gamma}_v$. From these relations follows the differential equation $\sigma + \tau_M\dot{\sigma} = \eta\dot{\gamma}$, where $\tau_M = \eta/G_0$ [8]. The general solution is $\sigma(t) = (\eta/\tau_M)\int_0^\infty \dot{\gamma}\exp(-s/\tau_M)ds$, which expresses the stress as a linear response on the time history of the strain rate with an exponentially decaying response function $G(s) = (\eta/\tau_M)\exp(-s/\tau_M)$. From the differential equation, we find a viscous response $\sigma \approx \eta\dot{\gamma}$, for variations on time scales $\ll \tau_M$, and an elastic response $\sigma = G_0\gamma$, for scales $\gg \tau_M$. For oscillations at frequency ω, we have a complex Young's modulus $G^*(\omega)$, with phase angle δ given by $\tan\delta = 1/(\omega\tau)$; $\delta \ll 1$ corresponding to elastic and $\delta \approx \pi/2$ to a viscous response. The latter is used in measurements of viscoelastic responses [9]. This model implies a separation between the elastic and hydrodynamic scales given by the response time τ_M. The results presented in this Letter do not exhibit such a scale separation, as shown by the appearance of long-range memory effects in the transport of dust grains. In the present context, long-range memory means that the integral $\int_0^\infty G(s)$ diverges, as will be the case if $G(s)$ decays algebraically rather than exponentially. The origin of memory effects and elastic properties is the emergence of vortex structures on different spatial scales, inside which some stiffness (lattice order) is maintained.

The system studied is a large circular cluster of 600 dust grains in a monolayer configuration. The experiment is performed in a capacitively coupled radio-frequency discharge operated in argon at a pressure ~4 Pa and rf power of 19 W. Rough estimates of plasma parameters are $n_e \sim 10^9$ cm^{-3}, $T_e \sim 2$ eV, and $T_i \sim T_n \sim 0.03$ eV. Injected monodispersive melamine-formaldehyde spheres with diameter $2a = 7.2$ μm become negatively charged and levitate as a monolayer in the sheath above the lower electrode. The grains are confined radially by the potential created by a cavity of 6 cm radius machined into the lower electrode. The particles are illuminated by a horizontal laser sheet. 30 000 images are taken by a video camera at a sampling rate of 30 Hz and spatial resolution of 24 μm/pixel. The database consists, therefore, of 600 time series (particle coordinates) of 30 000 points each, sampled at time intervals $\delta t = 1/30$ s. The grain kinetic temperature $T_d \sim 0.15$ eV is obtained by extrapolating the grain displacements at the shortest resolved time scales. Even though very weak rotation of the monolayer is detected, all data analysis has been carried out in the rotating frame of reference. The cluster diameter is 16 mm with interparticle distance $\Delta \approx 0.6$ mm. The system exhibits a hexagonal spatial structure with some five- and sevenfold defects, in particular, at the perimeter (Fig. 1). The pair correlation function (inset) reveals spatial order over distances of a few Δ. The particles are mobile, however, and on larger spatial and temporal scales vortex flows can be observed, as depicted in Fig. 2.

From the binary collision approach [10], we estimate a momentum transfer rate in dust-dust collisions $\nu_{dd} \sim 17\ s^{-1}$ while for dust-neutral collisions $\nu_{nd} \sim 6\ s^{-1}$. There is also direct evidence in the particle tracking data which supports these estimates, since we observe of the order of 100% change of particle momentum over one sampling interval $\delta t = 1/30$ s. This indicates that momentum relaxation happens on scales of the order of δt or faster. The same is confirmed by the autocorrelation function of particle velocity, which decays from unity to 0.2 within δt. Thus, there is solid evidence that there is a time scale separation for momentum relaxation in dust-dust and dust-neutral interactions, and we can consider our system as "one component," i.e., dominated by mutual interparticle collisions. We note, however, that for phenomena occurring on long time scales, where we observe large scale fluid flows, ν_{nd} introduces energy dissipation and cannot be ignored.

Essential information about the particle dynamics can be obtained by statistical analysis of the tracking data. The analysis is performed on the cumulative sum $\xi_j = \sum_{i=1}^{j} \delta\xi_i$ of the azimuthal position displacement during the sampling interval δt, defined as $\delta\xi_i = r_i \delta\varphi_i$. Here r_i is the distance from the center of the cluster at a time $i\delta t$, and $\delta\varphi_i$ is the increment in the azimuthal angle from time $(i-1)\delta t$ to $i\delta t$. The choice of the quantity ξ_j is motivated by the fact that it is not limited by the boundary in contrast to the radial position r_j. The *variogram* of the process ξ_j is defined as $V(\tau) = (N - \tau/\delta t)^{-1} \sum_{j=1}^{N-\tau/\delta t} (\xi_{j+\tau/\delta t} - \xi_j)^2$; i.e., it is the variance of the probability distribution function (PDF) of azimuthal position increments $\Delta\xi_j(\tau) = \xi_{j+\tau/\delta t} - \xi_j$ over the time lag τ. An example of such a PDF for $\tau \sim 2$ s is shown in Fig. 3(a). The standard deviation $\sigma(\tau) \sim \sqrt{V(\tau)}$ grows with time as a power law $\sigma(\tau) \sim \tau^H$ and is plotted in Fig. 3(b). The Hurst exponent H lies in the range between 0 and 1. Our results indicate that $H \approx 0.84$ for $\tau \lesssim 10$ s and $H \approx 0.68$ for $\tau \gtrsim 10$ s. This second value persists up to the longest time scales accessible in our study, $\tau \sim 500$ s. The fact that H is larger than 0.5 implies that the transport is superdiffusive. For times τ shorter than $\sim$30 s, the PDF of azimuthal displacements is consistent with a stretched Gaussian distribution

$$P(\Delta\xi, \tau) = A(\tau) \exp[-B(\tau)|\Delta\xi|^{2\mu}] \qquad (1)$$

and time-varying coefficients $A(\tau)$ and $B(\tau)$. Figures 3(c) and 3(d) show that these are best fitted to inverse power law dependencies $A(\tau) \sim \tau^{-\zeta/\mu}$ and $B(\tau) \sim \tau^{-2\zeta}$ in two separate regimes for τ. From the PDF in Eq. (1), one finds the

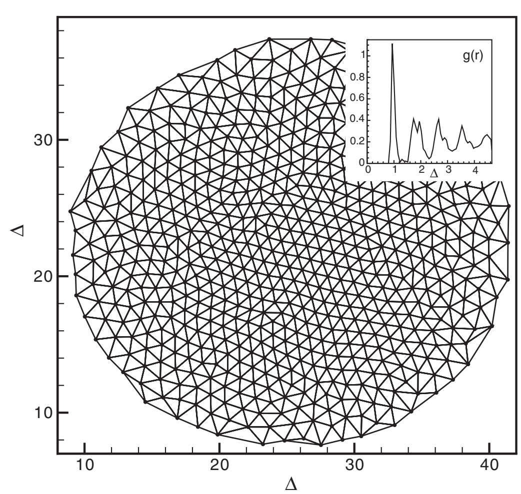

FIG. 1. A snapshot of the particle system showing strongly ordered structure and the pair correlation function (inset).

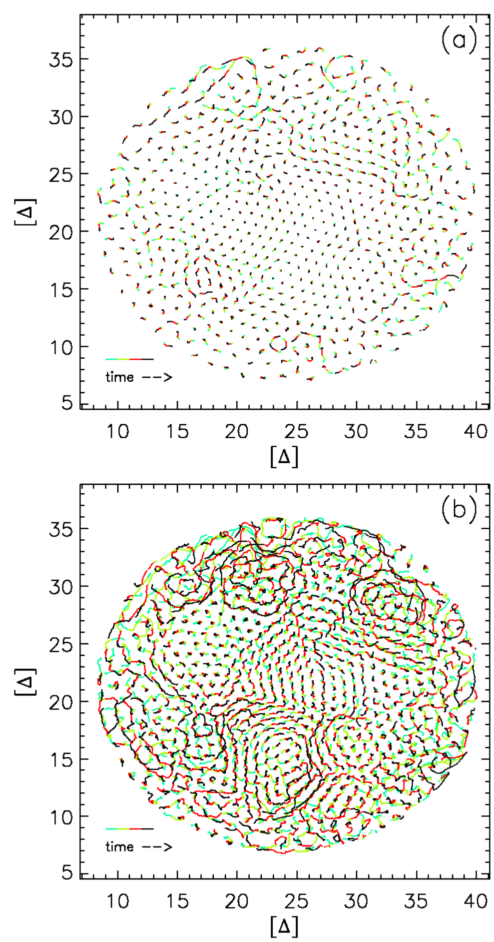

FIG. 2 (color online). Motion of particles during (a) $\sim$1 s and (b) $\sim$10 s.

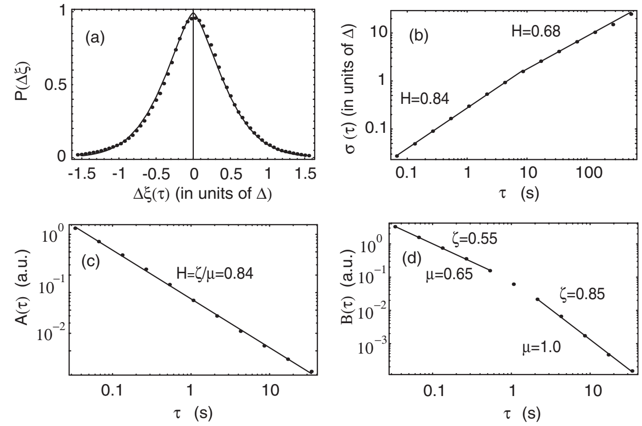

FIG. 3. (a) PDF of azimuthal position increments for time lag $\tau = 2$ s (dots). The nonlinear fit is the function given by Eq. (1) (solid curve). (b) Plot of $\sigma(\tau)$ [see text for definition]. (c),(d) Plot of $A(\tau)$ and $B(\tau)$ [see Eq. (1) for definition].

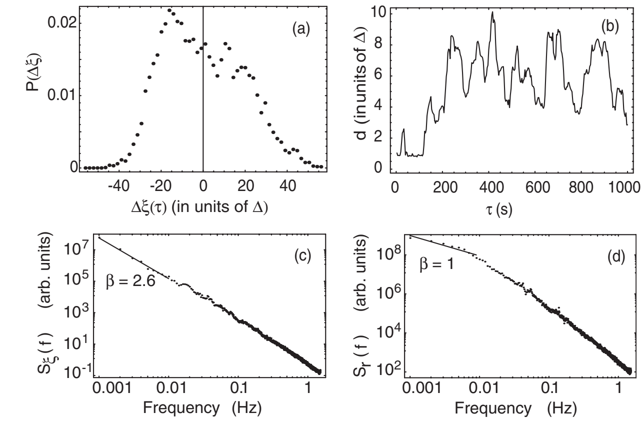

FIG. 4. (a) PDF of azimuthal position increments for time lag $\tau = 500$ s. (b) An example of relative diffusion (distance d between two particles as a function of time). (c),(d) Power spectra of azimuthal [$S_\xi(f) \sim f^{-\beta}$] and radial [$S_r(f) \sim f^{-\beta}$] displacements, with $\beta \sim 2.6$ and $\beta \sim 1$ for $f < 0.01$ Hz, respectively.

standard deviation $\sigma(\tau) \sim \tau^{\zeta/\mu}$, leading to $H = \zeta/\mu$. From Fig. 3(c), we find that $\zeta/\mu \approx 0.84$ for $\tau \lesssim 10$ s, consistent with the value found for H from the variogram analysis shown in Fig. 3(b). However, by plotting the coefficient $B(\tau)$ as in Fig. 3(d), we find that the exponents μ and ζ depend on τ, even though their ratio $H = \zeta/\mu \approx 0.84$ is almost constant up to $\tau \sim 10$ s. In fact, for the short times $\tau \lesssim 1$ s, we have $\zeta \approx 0.55$ and $\mu \approx 0.65$. These change to $\zeta \approx 0.84$ and $\mu \rightarrow 1$ for $\tau \gtrsim 1$ s. The fact that $\mu \rightarrow 1$ simply means that Eq. (1) tends to a Gaussian distribution on time scales $\tau > 1$ s. Since the Hurst exponent $H \approx 0.84$ is still larger than 0.5, the process is compatible with a persistent fractional Brownian motion (FBM), a self-affine, Gaussian stochastic process which exhibits long-range memory [11]. An ordinary Brownian motion appears as a summation of a Gaussian white noise process and an FBM as a summation of white noise filtered in favor of the low frequencies. The physics of this filtering is the emergence of cooperative hopping and vortex motions on time scales on which particles can move a distance Δ or more.

The more heavy-tailed PDF observed on shorter time scales indicates that the enhanced diffusion for $\tau \lesssim 1$ may be governed by mechanisms other than memory. Superdiffusion in the absence of long-range memory is normally ascribed to Lévy processes [12], for which the PDFs exhibit algebraic tails $P(\Delta\xi) \sim |\Delta\xi|^{-\mu}$. Algebraic tails are not observed here, but it is conceivable that the longer "Lévy flights" are prevented by some dissipative mechanism, e.g., neutral drag. This mechanism would then effectively truncate the tail of the PDF. The core of the Lévy PDF can be shown to converge to the stretched Gaussian distribution in Eq. (1) [13].

After approximately 30 s, the nearly Gaussian PDF develops an asymmetry, and at $\tau \approx 500$ s it has split up into two large and some smaller humps, as seen from Fig. 4(a). These humps show that the superdiffusive transport observed on time scales up to 30 s is replaced by advection of subpopulations of particles trapped in vortices of varying size. For the results $P(\Delta\xi, \tau)$ presented in Figs. 3, 4(a), 4(c), and 4(d), the study was based on a sample of 200 particles initially grouped together in one region of the cluster. The majority of these particles belong to the "interior," so on short time scale $\tau < 30$ s the statistics reflects typical interior behavior. In an edge boundary layer a few interparticle distances wide, the crystal bonds are weaker and the grains move more or less like in the liquid state. This fluid motion in the edge contributes to the long-time statistics presented in Fig. 4, but plots of individual trajectories show that the main transport of grains occurs in the interior, so the flow picture emerging from the statistical analysis on long time scales is also characteristic for the dynamics in the interior of the cluster.

Additional support for the vortex flow interpretation can be found from the relative diffusion of the particles. Figure 4(b) shows the evolution of the distance d between two particles that start out as nearest neighbors ($d \approx \Delta$ at $t = 0$). On time scales of the order of some tens of seconds, one observes jumps in d of approximately one Δ, sometimes as single slip events but more often as a continuous chain of several consecutive slips. On longer time scales, these cooperative events form oscillatory motions over a broad spectrum of frequencies corresponding to trapping of the two particles in vortices of varying size up to the size of the system. The same can be deduced from the power spectrum of azimuthal displacements $S_\xi(f) \sim f^{-\beta}$, with $\beta \sim 2.6$ for frequencies below 0.1 Hz [Fig. 4(c)]. The relation to the Hurst exponent is $H = (\beta - 1)/2$, giving $H \approx 0.8$ for $\tau > 10$ s, somewhat higher than found from the variogram in Fig. 3(b). The latter estimate is more reliable, since it is known that the variogram underestimates the true self-affinity parameter for strongly superdiffusive fractional Brownian motions [11]. The power spectrum of radial displacements $S_r(f) \sim f^{-\beta}$ has $\beta \sim 1$ for the frequency range lower than 0.01 kHz [Fig. 4(d)]. The corresponding variogram (not shown) departs from the variogram for the azimuthal displacement for $\tau > 10$ s and becomes flat for $\tau > 100$ s. This is due to the limitation of the radial displacement set by the finite cluster size. The spectral index $\beta = 1$ ($H_a = 0$) on the hydrodynamic time scale $\tau > 100$ s corresponds to an antipersistent process at the transition between a ''noise'' process (stationary) and a ''motion'' (nonstationary), also called ''pink noise'' or ''$1/f$ noise'' [11].

The idea that the grain dynamics can be described as a viscoelastic flow derives mainly from figures of the flow patterns such as Fig. 2 (and study of the movies [14]) and from the results of the statistical studies presented in Figs. 3 and 4. In Fig. 2, one observes large domains where motion on time scales <30 s occurs along the three principal directions (crystal surfaces) in a hexagonal crystal structure, and yet the collective motion has the character of rotation of this domain (a vortex). The edges of the domains follow the crystal surfaces, and in their interior the crystal order is maintained; i.e., on this time scale, grains do not interchange positions in the lattice. Geometric constraints then imply a certain deformation (strain) of the domain as it rotates, and the resulting stress corresponds to elastic behavior. On the border between domains, development of crystal defects allows grains to slip relative to each other, and on these boundaries the stress depends on the strain *rate* rather than the strain itself; i.e., the motion is viscous. If there were one characteristic vortex size and turnover time, we would have two distinct scales: elastic on small scales and viscous on large. However, since Fig. 2, as well as Figs. 4(a) and 4(b) indicate a wide range of vortex sizes, it is not possible to separate these scales, and the statistically averaged stress-strain relation may exhibit a ''long-range'' response function $G(s) \sim s^{-\alpha}$, with $\alpha \leq 1$. On time scales below a few dust-neutral relaxation times $\nu_{\rm dn}^{-1}$, the transport seems to be governed by Lévy statistics where the algebraic tails of the distribution are truncated, possibly due to the effects of collisions with neutrals. On longer time scales, the transport is superdiffusive due to memory effects. Long-range memory means that the long scales are more strongly represented in the spectrum, indicating that vortex motions on varying scales are responsible for this memory-based superdiffusion.

For a complete description of the system, we also need to understand the mechanisms responsible for energization of the dust grains and driving the vortical flow against the neutral gas friction. As has been pointed out by Zhakhovskii *et al.* [15], the fact that the particle charge depends on the spatial coordinates implies that the energy of particles in external electric fields is not conserved. This is because the external electrostatic force is no longer potential. The sign of the work done on a particle over a closed path depends on the direction of motion along the path; hence, the formation of vortical patterns is consistent with a balance between positive energy gain over the cycle and the frictional loss due to neutral drag, as recently suggested in Refs. [16,17]. Therefore, one of the possible driving mechanisms of the observed vortex formation can be the charge inhomogeneity across the layer. Another energy source for horizontal particle motion (especially at the short time scales) might arise from the interaction with the wakes which are formed under each particle by the downward streaming ions (see, e.g., [18]). It has been shown that the wakes can cause horizontal acceleration of a single particle suspended *under* the main layer [19], and one cannot exclude the possibility that a similar mechanism affects particles in the layer.

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Subdiffusive Lévy flights in quantum nonlinear Schrödinger lattices with algebraic power nonlinearity

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We report a theoretical result concerning the dynamics of an initially localized wave packet in quantum nonlinear Schrödinger lattices with a disordered potential. A class of nonlinear lattices with subquadratic power nonlinearity is considered. We show that there exists a parameter range for which an initially localized wave packet can spread along the lattice to unlimited distances, but the phenomenon is purely quantum and is hindered in the corresponding classical lattices. The mechanism for this spreading is moreover very peculiar and assumes that the components of the wave field may form coupled states by tunneling under the topological barriers caused by multiple discontinuities in the operator space. Then these coupled states thought of as quasiparticle states can propagate to long distances on Lévy flights with a distribution of waiting times. The overall process is subdiffusive and occurs as a competition between long-distance jumps of the quasiparticle states, on the one hand, and long-time trapping phenomena mediated by clustering of unstable modes in wave number space, on the other hand. The kinetic description of the transport, discussed in this work, is based on fractional-derivative equations allowing for both (i) non-Markovianity of the spreading process as a result of attractive interaction among the unstable modes; this interaction is then described in terms of the familiar Lennard-Jones potential; and (ii) the effect of long-range correlations in wave number space tending to introduce fast channels for the transport, the so-called "stripes." We argue that the notion of stripes is key to understand the topological constraints behind the quantum spreading, and we involve the idea of stripy ordering to obtain self-consistently the parameters of the associated waiting-time and jump-length distributions. Finally, we predict the asymptotic laws for quantum transport and show that the relevant parameter determining these laws is the exponent of the power law defining the type of the nonlinearity. The results presented here shed light on the origin of Lévy flights in quantum nonlinear lattices with disorder.

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I. INTRODUCTION

Waves in random systems cannot readily propagate to long distances: scattered by structural inhomogeneities on many spatial scales, they tend to form multiple standing waves at high disorder, and this effectively confines the wave process within a spatially restricted domain. The phenomenon—predicted by Anderson in 1958 [1] and extensively studied ever since—has come to be known as the Anderson localization and occurs for any type of wave process, classical or quantum.

A continued interest in the phenomena of Anderson localization was due to the direct experimental observation of the Anderson localization of visible light [2] and the measurement of the critical exponent of scaling theory of the localization transition [3]. More recently, there has been a stream of literature stimulated by Pikovsky and Shepelyansky [4,5] that sought to demonstrate that the Anderson localization in random systems could be destroyed by a weak nonlinearity and that the phenomenon is thresholded in that there exists a critical strength of nonlinear interaction such that above this strength the nonlinear field can propagate across the lattice to infinitely long distances, and is Anderson localized despite these nonlinearities otherwise.

Theoretically, the destruction of Anderson localization in nonlinear lattices has been studied in the fashion of the Gross-Pitaevskii (i.e., nonlinear Schrödinger) equation with disordered potential [4–13]. A modified perturbation theory with regard to the strength of the nonlinear term has been developed [6,9], and extensive numerical simulations have been carried out [7–9]. A subdiffusive scaling for the onset spreading has been introduced and numerically measured [5,7]. A nonperturbative approach to the nonlinear Anderson problem has been developed based on topological approximations, using random walks and the concept of critical percolation on a Cayley tree [12–15]. The subject has attracted additional interest recently in view of its extension to quantum dynamics [16,17] and the suggestion—motivated by Fermi's golden rule—that the loss of localization in the quantum domain could be not thresholded [18].

Our purpose here is to describe the delocalizing effect of subquadratic power nonlinearity on quantum dynamics of a lattice gas in nonlinear Schrödinger lattices with disorder. A background for this consists in the following. (i) It has been shown [12–15] based on a classical analysis that a power nonlinearity of the Ginzburg-Landau type (i.e., quadratic power nonlinearity) played a very special role in classical

Self-consistent model of the plasma staircase and nonlinear Schrödinger equation with subquadratic power nonlinearity

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A new basis has been found for the theory of self-organization of transport avalanches and jet zonal flows in L-mode tokamak plasma, the so-called "plasma staircase" [Dif-Pradalier *et al.*, Phys. Rev. E **82**, 025401(R) (2010)]. The jet zonal flows are considered as a wave packet of coupled nonlinear oscillators characterized by a complex time- and wave-number-dependent wave function; in a mean-field approximation this function is argued to obey a discrete nonlinear Schrödinger equation with subquadratic power nonlinearity. It is shown that the subquadratic power leads directly to a white Lévy noise, and to a Lévy fractional Fokker-Planck equation for radial transport of test particles (via wave-particle interactions). In a self-consistent description the avalanches, which are driven by the white Lévy noise, interact with the jet zonal flows, which form a system of semipermeable barriers to radial transport. We argue that the plasma staircase saturates at a state of marginal stability, in whose vicinity the avalanches undergo an ever-pursuing localization-delocalization transition. At the transition point, the event-size distribution of the avalanches is found to be a power law $w_\tau(\Delta n) \sim \Delta n^{-\tau}$, with the drop-off exponent $\tau = (\sqrt{17}+1)/2 \simeq 2.56$. This value is an exact result of the self-consistent model. The edge behavior bears signatures enabling to associate it with the dynamics of a self-organized critical (SOC) state. At the same time the critical exponents, pertaining to this state, are found to be inconsistent with classic models of avalanche transport based on sand piles and their generalizations, suggesting that the coupled avalanche-jet zonal flow system operates on different organizing principles. The results obtained have been validated in a numerical simulation of the plasma staircase using flux-driven gyrokinetic code for L-mode Tore-Supra plasma.

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I. INTRODUCTION

Recently, due to the high-resolution, ultrafast sweeping reflectometry schemes employed in the fusion research, there has been increasing attention both theoretically and experimentally on the issues related with the propensity of toroidally confined L-mode plasma to spontaneously generate microbarriers to radial transport as a result of plasma self-organization. Often such barriers are found to occur in quasiregular patterns of highly concentrated, multiple jet zonal flows interspersed with broader regions of turbulent (typically, avalanching) transport [1–4]. The phenomenon, illustrated numerically in Fig. 1 with the aid of a flux-driven gyrokinetic code [5], has come to be known as the plasma staircase and was so named [1] after its celebrated planetary analog [6].

The physics of the plasma staircase is of interest from both a fundamental scientific perspective and for the practical realization of fusion energy. From a scientific perspective, the nonlinear dynamics of the plasma staircase occupies an interesting niche where microscale and mesoscale nonlinearities can appear on an equal footing [3,4]. From a practical perspective, the periodic dynamical patterning due to the plasma staircase offers a unique environment to control the avalanche activity by fine tuning the shape and the radial positions of the barriers [4,7]. These practical aspects are dictated by the understanding that the avalanche transport may have a deteriorating effect on the confinement properties of thermal plasma and charged fusion products [8,9], while significant losses could be detrimental. It is therefore a crucial issue to understand the behavior of the coupled staircase-avalanching system and the way the avalanches may be contained within the steps of the barriers.

Although the plasma staircase is a relatively new topic for fusion, it already enjoys an exciting history behind: The phenomenon was discovered experimentally [2] on the Tore Supra tokamak following its very precise theoretical prediction in Ref. [1], a rare, classic circumstance when the discovery is made *au bout de sa plume*, if the celebrated art phrase due to Arago [10] is appropriate here. At the time this paper was being written, the natural tendency of L-mode

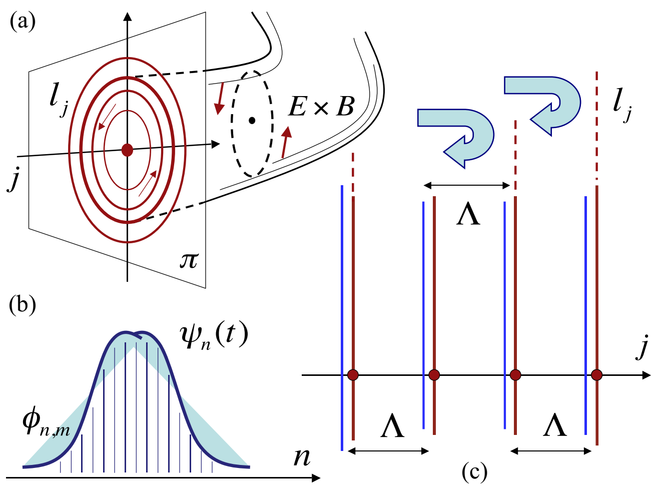


FIG. 2. (a) A birds-eye view of the jet zonal flows and their poloidal cross section (poloidal plane is marked with the letter π). (b) A schematic representation of the staircase wave function $\psi_n(t) = \psi(n,t)$. Vertical lines mimic the eigenfunctions of the linear problem $\phi_{n,m}$. (c) The discrete structure of jet zonal flows at the short view and the definition of the position coordinate j. The distance between the jets (velocity extrema) is Λ and is estimated as the Rhines length for electrostatic drift-wave turbulence, i.e., $\Lambda \sim \Lambda_{\rm Rh}$ (Sec. IV D). The radial position is tracked by the variable n, which also characterizes the energy spectrum of the staircase dynamical system under the NLSE approximation. Top: U-turn arrows illustrate the avalanches confined between the staircase steps.

packet being broad enough in that it contains a large number of the individual jets. The coupling between the jets is provided by their nonlinear interaction, which is mediated by the avalanches. In a self-regulating, nonlinear plasma system, that would be a rather efficient mechanism since the avalanches, absorbed by the transport barriers, deliver momentum to the poloidal flows (via the turbulent Reynolds stress), which in turn enhances the strength of the barriers [33].

A. Description of the model

To characterize, from a most general perspective, the nonlinear dynamics of a wave packet of coupled nonlinear oscillators, the jet zonal flows, one might invoke the analytical scheme of the nonlinear Schrödinger equation, or NLSE [34,35], also known as the Gross-Pitaevskii equation [36,37]. The time-dependent Gross-Pitaevskii equation describes the dynamics of initially trapped Bose-Einstein condensates and is shown to be an exact equation in the dilute limit [38,39]. For many-body bosonic systems, the NLSE is a mean-field approximation where the term proportional to the probability density $|\psi|^2$ represents the interaction between the atoms.

Next, we argue (and confirm through results) that the self-organization phenomena pertaining to the plasma staircase require a modified form of NLSE in which the probability density $|\psi|^2$ is replaced by a subquadratic power nonlinearity $|\psi|^{2s}$, where $0 < s < 1$ is a power exponent and tunes the nonlinear interaction mechanism. This modified form of NLSE has been considered in Ref. [28] for the destruction of Anderson localization in quantum nonlinear Schrödinger lattices with disorder.

In the nonlinear Anderson problem the subquadratic nonlinearity arises because the nonlinear interactions among the waves might be subject to a competing nonlocal ordering (such as, for instance, the stripy ordering [40,41], etc.), leading the constituent linear waves to interfere with themselves [26,28]. This destructive self-interference might be either complete, eliminating the dependence on the modulus field (for $s = 0$), or partial (for $0 < s < 1$), and is parametrized by the subquadratic power $2s < 2$. No competing ordering (no self-interference) is assumed to take place for the quadratic power nonlinearity, with $s = 1$. In magnetically confined fusion plasma, a structural disorder similar to the disorder in the Anderson problem might occur thanks to the presence of a low-frequency, electrostatic micro-turbulence (e.g., Refs. [42–45], Ref. [46] for review); in the meantime, the competing nonlocal ordering could be associated with spontaneous occurrence of the jet zonal flows [47] or staircase self-organization [2–4], suggesting a similar dynamical description. Note that the drift waves are simultaneously a source for the disorder and the driving mechanism for the zonal flows. With these implications in mind, we introduce a discrete NLSE of the form

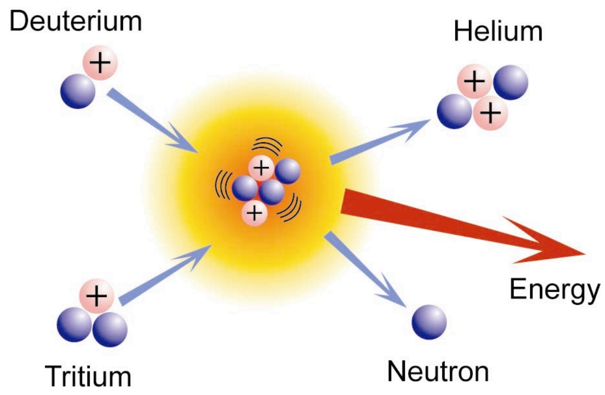
$$i\hbar\frac{\partial\psi_n}{\partial t} = E_n\psi_n + \beta|\psi_n|^{2s}\psi_n + V(\psi_{n+1} + \psi_{n-1}), \quad (1)$$

where $\psi_n = \psi(n,t)$ is a complex wave function and describes the plasma staircase as a compound system of coupled nonlinear oscillators; n is the discrete coordinate and is associated with the radial direction in a tokamak; β characterizes the strength of nonlinearity (below for definiteness $\beta > 0$); $0 < s < 1$ absorbs the effect of competing ordering on wave-wave interactions; V is transition matrix element; E_n are onsite energies; and the total probability is normalized to unity: $\sum_n |\psi_n|^2 = 1$. In what follows, $\hbar = 1$ for simplicity; thus, the energy coincides with the frequency. For $\beta \to 0$, the staircase decays into a set of loosely connected eigenstates, i.e., (almost) noninteracting jet flows, whose eigenfunctions are exponentially localized, the localization length being much smaller than the spacing between the jets. Note that we have introduced n instead of j to be the position coordinate in the NLSE model [see Figs. 2(b) and 2(c)]. A reason for that is that n bears a somewhat different implication in that it directly characterizes the energy spectrum of the staircase system under the NLSE approximation. We assume that the spectrum E_n is discrete and dense, for the positions of the jet zonal flows must correspond to rational values of the tokamak safety factor. Note that the safety factor [48] is usually a function of radius, implying that the energy spectrum E_n might be actually very broad, consistently with the above assumptions.

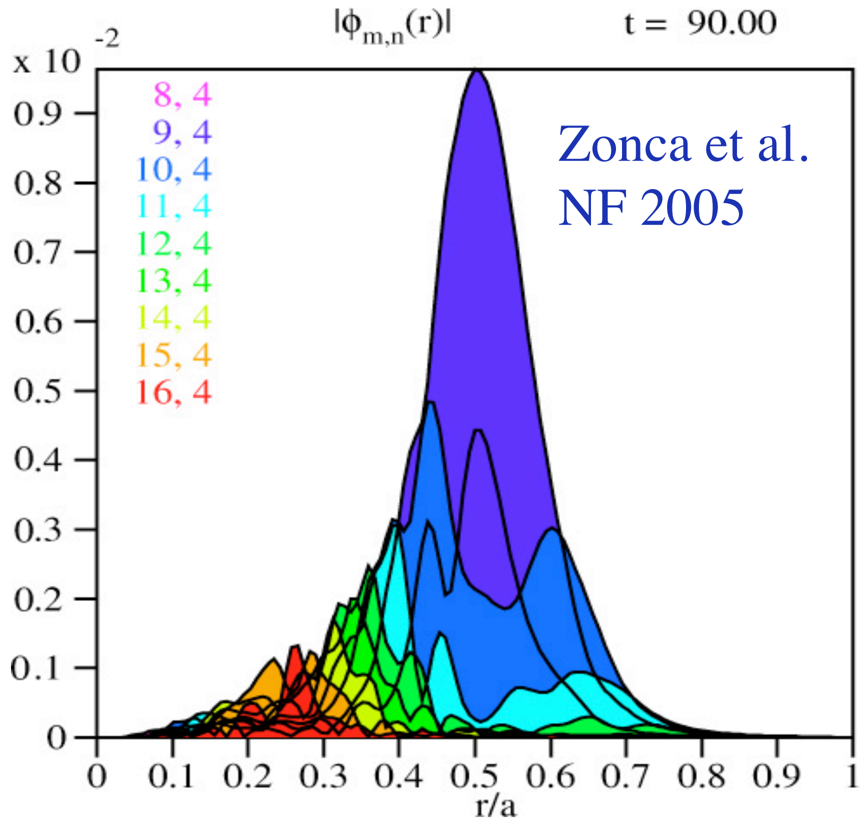
The background theory for NLSE (1) refers to wave processes with competition between dispersion, randomness, and nonlinearity (e.g., Refs. [49–55]). A large body of work promoting a reduced equation with quadratic power nonlinearity ($s = 1$) is documented in Refs. [56–65]. A generalization to subquadratic powers, with $s < 1$, was formulated in Refs. [26–28]. Superquadratic nonlinearities have been considered in Refs. [25–27]. Formally, the model in Eq. (1) coincides with the NLSE model introduced in Ref. [28], but

For the Lévy flights to occur (PRE 2021)

- Chaotic dynamics (EPM: probably YES)
- Competing non-local ordering (s < 1; suppressed by EPM: s = 1)
- In vicinity of attracting stable state (EPM: probably NOT)



Need for stationary white-noise process, otherwise is overly driven (convective amplification)



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Lévy flights on a comb and the plasma staircase

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We formulate the problem of confined Lévy flight on a comb. The comb represents a sawtoothlike potential field $V(x)$, with the asymmetric teeth favoring net transport in a preferred direction. The shape effect is modeled as a power-law dependence $V(x) \propto |\Delta x|^n$ within the sawtooth period, followed by an abrupt drop-off to zero, after which the initial power-law dependence is reset. It is found that the Lévy flights will be confined in the sense of generalized central limit theorem if (i) the spacing between the teeth is sufficiently broad, and (ii) $n > 4 - \mu$, where μ is the fractal dimension of the flights. In particular, for the Cauchy flights ($\mu = 1$), $n > 3$. The study is motivated by recent observations of localization-delocalization of transport avalanches in banded flows in the Tore Supra tokamak and is intended to devise a theory basis to explain the observed phenomenology.

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I. INTRODUCTION

In recent investigations of zonal flow phenomena in magnetized plasma by means of high-resolution ultrafast-sweeping X-mode reflectometry in the Tore Supra tokamak, spontaneous flow patterning into a quasiregular sequence of strong and lasting jets interspersed with broader regions of turbulent (typically, avalanching) transport has been observed [1–3]. The phenomenon was dubbed "plasma staircase" by analogy with its notorious planetary analog [4]. The plasma staircase has been referred to as an important self-organization phenomenon of the out-of-equilibrium plasma, which had pronounced effect on radial transport and the quality of confinement. Detailed analyses (both experimental and numerical based on gyrokinetic calculations) have identified the plasma staircase as a weakly collisional, mesoscale [5] dynamical structure near the state of marginal stability of the low confinement mode plasma [2,3].

The comprehension of the plasma staircase [1] has both fundamental and practical significance. From a scientific perspective, the plasma staircase represents a fascinating dynamical system in which kinetic and fluid nonlinearities may operate on an equal footing. In the practical perspective, the plasma staircase raises the important problem of avalanche-zonal flow interaction [2,3], which may be key to control the dynamic confinement conditions in magnetic fusion devices, tokamaks and stellarators. On top of this, the fact that a significant portion, if not a vast majority, of avalanches have been confined within the staircase steps [3] is by itself a challenge, since the plasma avalanches being spatially extended transport phenomena behave dynamically nonlocally, and their "localization" within a transport barrier is not at all obvious. Mathematically, this revives the long-standing problem of the *confined Lévy flight*, which has attracted attention in the literature previously (e.g., Refs. [6–11]).

In this paper, we adapt the general problem of confined Lévy flight [7,9] for staircase physics and show that the transport avalanches may be localized, if (i) the staircase jets are spatially separated, as they prove to be [2,3], and (ii) at each step of the staircase the gradients are sharp enough in that the potential function grows faster with distance than a certain critical dependence (cubic when modeled by a power-law). If the growth is slower than this, then the avalanches are *not* localized in that there is an important probability of finding the Lévy flyer outside the transport barrier. More so, we find that in the confinement domain there may occur at least three different types of avalanches, which we call, respectively, *white swans*, *black swans* [12], and *dragon kings* [13], and that the white swans may "mutate" into the black swan species past the intermediate *gray-swan* family found at the point of cubic dependence. This gives rise to some features of bifurcation, which might be identifiable in the experiment. This observation opens a new perspective on "smart" plasma diagnostics in tokamaks using plasma self-organization [1–3].

The paper is organized as follows. In Sec. II, we introduce an idealized transport model, which we arguably name *Lévy flights on a comb*, and which is motivated by the challenges discussed above. The model, which is derived in Sec. II B using the idea of transition probability in reciprocal space [14], is intended to mirror the observed behaviors [1–3] and, most importantly, provide a practical criterion for the phenomena of localization-delocalization of avalanches in the presence of zonal flows. We discuss the various aspects of this model in Secs. III and IV, which focus on, respectively, space scale separation issues and the size distribution of avalanches. The latter is shown to be inverse power-law for both the white and black swans, but with different drop-off exponents, making it possible to differentiate between the species. We conclude the paper in Sec. V with a few remarks.

II. THE MODEL

We represent the plasma staircase as a periodic lattice, a comb, looking along the coordinate x; the latter represents by the turbulence take energy from the turbulence, meaning that their driving mechanism is diminished, and they may be decaying due to classical or neoclassical collisional damping (e.g., Refs. [16,50]). The process opens a possibility that some avalanches escape the confinement domain during the barrier depression periods, giving rise to sporadic bursts of large-scale transport well above the staircase's parapet. This type of occasionally strong transport events being virtually insensitive to the underlying flow and stress organization has been found in the GYSELA simulations [2,3], and their statistical weight has been assessed to be about a percentile of all avalanche events observed across the staircase.

If one is a traditionalist, and wants to remain with the Fokker-Planck model in Eqs. (1) and (2), then one might readily assess the statistical case of unconfined avalanches as follows. In the basic kinetic equations, one neglects both the Gaussian and the potential force terms, as well as the sink term $\hat{S}_{-}[f(x)]$, and only keeps the nonstationary term against the Lévy term. The net result is that (i) there is no steady state solution, contrary to the confined transport case; and (ii) the probability density, which is time dependent, behaves asymptotically as a power-law $f(x,t) \sim K_{\mu} t / x^{1+\mu}$. Due to this property, the mean squared displacement diverges, i.e., $\langle x^2(t)\rangle \to +\infty$, which is typical for free Lévy flights. In view of this divergence, the size distribution of unconfined avalanches is obtained as the corresponding jump length distribution [23]. The latter is given by Eq. (15), yielding, for $\Delta s \gg \ell$, $\Delta s \gg \sqrt{D/q}$,

$$w(\Delta s) \propto \Delta s^{-(1+\mu)}. \tag{32}$$

The scaling in Eq. (32) is confirmed by tuning n to its borderline value $n = 3$ in $w(\Delta s) \propto \Delta s^{-(n+\mu-2)}$, as is intimated by Eq. (29) above.

Let us christen our avalanches. Inspired by the mathematical elegance of the confined Lévy flight, we baptize the avalanches caught in-between the staircase steps *white swans*. The term is intended to contrast the other population of bursty transport events, the *black swans*, which are the avalanches escaping the confinement system during the low barrier phase. The name *black swan* is borrowed from the Taleb's book [12]; where, it has been introduced to describe an unexpected catastrophic event catching us off-guard. Note that the size distributions of the power-law type appear for both the white and black swans, but with different drop-off exponents, so that for $n > 3$ the black-swan distribution is *always* flatter (in its habitat) than the corresponding white-swan distribution (see Fig. 2).

The occurrence of the black-swan family gives rise to a characteristic "bump" in the $w(\Delta s)$ dependence, which is located around $\Delta s \sim \Lambda$. Given the space scale separation condition $\Lambda \gg \sqrt{D/q}$, the position of this bump is well beyond the exponential core region (see Fig. 2). One sees that the resulting $w(\Delta s)$ dependence, which embraces both the white- and black-swan populations, will be *bimodal* in that it has a second maximum near $\Delta s \sim \Lambda$.

Note, also, that the white swans go extinct beyond the staircase spacing distance $\sim\Lambda$, that is, the areas of the white- and black-swan dominance are essentially different (except for the narrow overlap region around $\sim\Lambda$). This finding is peculiar and says the probabilities of the black-swan events *cannot* be predicted by interpolating the white-swan counterpart (if it exists) to longer sizes.

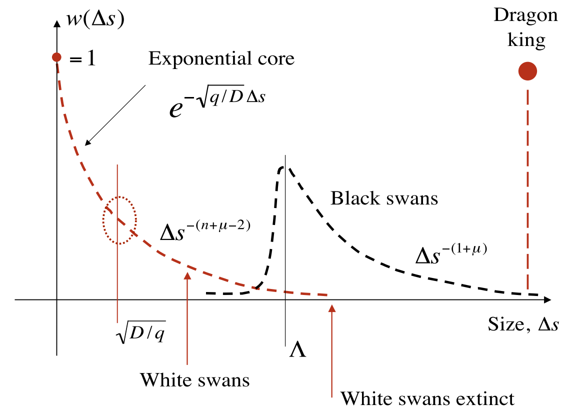

FIG. 2. The coexistence between the white- and black-swan families of avalanches for $n > 3$. The occurrence of the black-swan family gives rise to a characteristic "bump" in the $w(\Delta s)$ dependence around $\Delta s \sim \Lambda$, lying far off the exponential core region (i.e., the property of bimodelity). The dragon-king avalanches being singular transport events are shown as a fat dot at the upper-right corner dominating the scene.

The respective drop-off exponents for the white and black swans would only coincide for the borderline case $n = 3$, for which all the swans stick together to form one single family, with the unique size distribution $w(\Delta s) \propto \Delta s^{-(1+\mu)}$. Arguably, one might refer to this borderline case as *gray swans*, as they serve as the missing bond between the two main species, the white and black swans. Because $\mu < 2$, the gray swans correspond to nonlocalized avalanches.

For $n < 3$ (but still larger than 2, see Sec. II), we expect the white swans to completely change their color and "mutate" (past the intermediate gray-swan phase) into one single family of the black-swan type populating the entire staircase (see Fig. 3), with the unique size distribution $w(\Delta s) \propto \Delta s^{-(n+\mu-2)}$. The latter distribution turns out to be flatter than the jump length distribution for free Lévy flights as of Eq. (32) above (this is because the n value is now smaller than 3), implying that the asymmetric-teeth effect enhancing the transport in radial direction has become

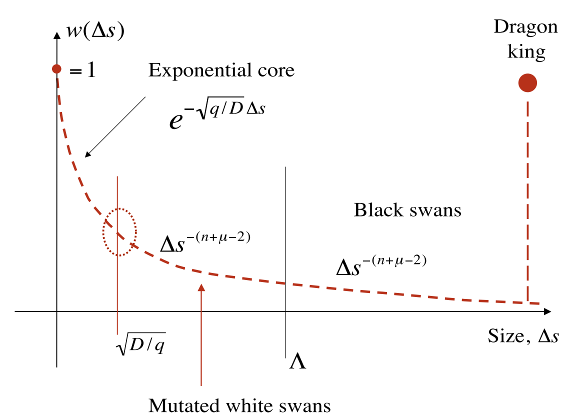

FIG. 3. Same situation, but for $2 < n < 3$. The regime with $n = 3$ is the borderline case, for which the white-swan family "mutates" into one extended black-swan family past the gray-swan species. The bimodality of the $w(\Delta s)$ dependence (see Fig. 2 above) is naturally lost in this case.

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Frontiers

Black swans, extreme risks, and the e-pile model of self-organized criticality

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ABSTRACT

A new model of self-organized criticality (SOC) is described which differs from ordinary sand-pile SOC models in that it uses electrically charged particles of different kinds to propagate activities and to generate the critical state. The model, arguably called the *e-pile* model (as an alternative to *sand-pile*, with "e-" standing for "electric charge"), is motivated by the problem of dielectric relaxation in self-asembling random lattices with disorder under the action of electrostatic forces, but in principle it may be applied to different SOC processes with the random injection scheme and charitable lattice-redistribution rule. We show that the critical state is that of self-evolving random percolation clusters at the edge of percolation and is also different from known "self-organized" versions of the percolation problem based on the directed percolation. A set of critical exponents is obtained based on the random walks, using the Kramers-Kronig relation and the formalism of frequency-dependent complex conductivity. The relaxation of a supercritical system to SOC is shown to obey the Mittag-Leffler pattern and fractional relaxation equation, with a broad distribution of durations of relaxation events. A peculiar feature of the e-pile system is that it is characterized by an intrinsic bias disregarding a vast majority of incipient relaxation events (by killing them "in their egg"), while it also favors other events that may, then, grow into extremely large sizes. We use this observation to explain the phenomenon of the "black swan," by which one means a family of rare, large events, whose sudden occurrence is very difficult to predict, despite the important impact this type of event may have over the entire system. The event-size distribution of the black swans is found to be a power law, with the drop-off exponent being sensitive to the complexity features of the underlying percolation clusters. On the basis of this observation, we show that the black swans are less probable in more complex environments, and we use this argument to explain why driven, dissipative systems would develop complex structures as a result of dynamical evolution. The e-pile model reveals the upper critical dimension $d_c = 6$ for which a crossover to mean-field SOC is found. In six and higher dimensions, the probability density to observe a black swan of a given size behaves as the inverse cube of the size. Below d_c the decay is always slower than the inverse-cube and is flatter at lower dimensions. The implications of the mean-field law in country risk assessment, business, and finance are discussed. Finally, we show that the occurrence of the power-law tails is limited to the rate of the driving, and that for too high a rate a different branch of extreme events may result, with the defining features enabling to associate this branch with the phenomenon of dragon kings.

1. Introduction

"Black swan" [1] is an art name for an abrupt, singular, large-magnitude event, whose sudden occurrence is very difficult to predict, despite the potentially disastrous effect it may imply. The phenomenon is of interest from both a fundamental scientific perspective and for the practical implementation of hazard assessment. From a scientific perspective, large-magnitude events attract considerable attention because they reveal the underlying, often hidden, organizing principles behind the nonlinear dynamics of systems with many interacting degrees of freedom [2–7]. From a practical perspective, the disproportional role of high-profile, singular events poses a challenge to our civilization, with important

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