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TSVV3 – Task 7: Status and plans EBC code

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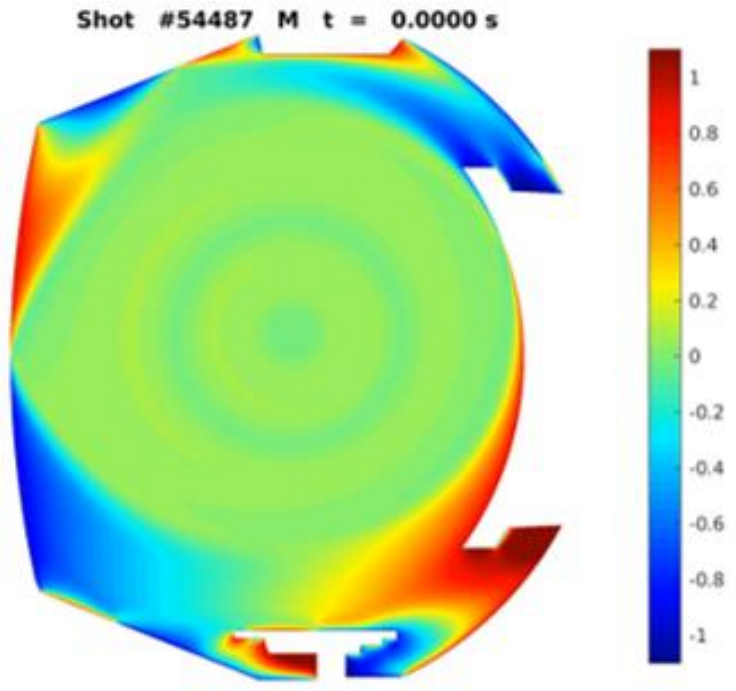
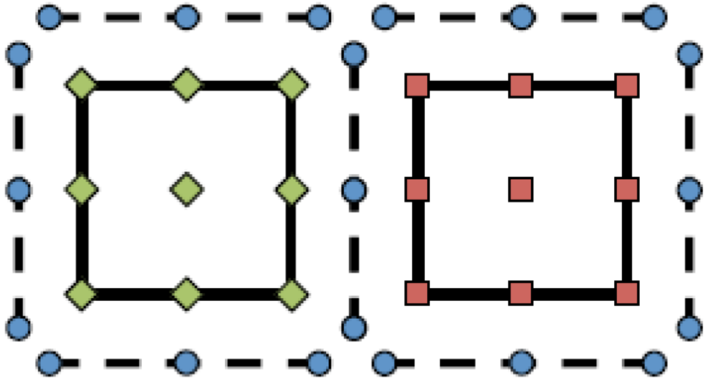
Motivation

- Why HDG-approach?
  - Allows detailed geometrical description of the wall
  - Grid flexibility
  - High order, p-adaptivity
  - Highly parallelizable
  - Cheaper than standard implicit FEM

- Drawbacks HDG-approach?
  - High order $\Rightarrow$ badly conditioned matrices





2D interchange model

- Targeted first model

$$\partial_t n + \nabla \cdot (n(\boldsymbol{u}_E - T_e \boldsymbol{K}) - D_\perp \nabla_\perp n) = S_n$$

$$\partial_t n_n + \nabla \cdot \left(\frac{n_{n,eq}}{n} (n\boldsymbol{u}_E + p_i \boldsymbol{K}) - \frac{p_i}{B} \boldsymbol{b} \times \nabla_\perp \left(\frac{n_{n,eq}}{n} \right) - \frac{1}{n(K_{cx} + K_i)} \nabla_\perp (T_i n_n) \right) = S_{n_n}$$

$$\partial_t p_e + \frac{5}{3} \nabla \cdot \left(p_e \boldsymbol{u}_E - \frac{p_e^2}{n} \boldsymbol{K} - \frac{p_e}{n} D_\perp \nabla_\perp n \right) - \frac{2}{3} \nabla \cdot (\chi_{\perp,e} n \nabla_\perp T_e) = S_{p_e}$$

$$\partial_t p_i + \frac{5}{3} \nabla \cdot \left(p_i \boldsymbol{u}_E + \frac{p_i^2}{n} \boldsymbol{K} - \frac{p_i}{n} D_\perp \nabla_\perp n \right) - \frac{2}{3} \nabla \cdot (\chi_{\perp,i} n \nabla_\perp T_i) = S_{p_i}$$

$$\partial_t W + \nabla \cdot (W (\boldsymbol{u}_E + T_i \boldsymbol{K}) - \nu \nabla_\perp W) + \nabla \cdot ((p_i + p_e + p_n) \boldsymbol{K}) = S_W$$

with

$$W = \nabla \cdot \left(\frac{\nabla_\perp \phi}{B^2} + \frac{\nabla_\perp p_i}{B^2} \right)$$

$$\boldsymbol{u}_E = \frac{\boldsymbol{b} \times \nabla \phi}{B}, \quad \boldsymbol{K} = 2 \frac{\boldsymbol{b} \times \nabla \ln B}{B}, \quad n_{e,eq} = \frac{n_n K_{cx} + n K_r}{K_{cx} + K_i}$$

2D interchange model

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$$\partial_t n + \nabla \cdot (n(\boldsymbol{u}_E - T_e \boldsymbol{K}) - D_\perp \nabla_\perp n) = S_n$$

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$$\partial_t W + \nabla \cdot (W (\boldsymbol{u}_E + T_i \boldsymbol{K}) - \nu \nabla_\perp W) + \nabla \cdot ((p_i + p_e + p_n) \boldsymbol{K}) = S_W$$

$\Rightarrow$ set of coupled advection-diffusion-reaction equations for $\boldsymbol{U} = \{\boldsymbol{U}_1, ..., \boldsymbol{U}_5\}^T = \{n, n_\mathrm{n}, p_\mathrm{e}, p_\mathrm{i}, W\}^T$

$$\partial_t \boldsymbol{U} + \boldsymbol{\nabla} \cdot (\boldsymbol{C}(\boldsymbol{U}) + \boldsymbol{K}(\boldsymbol{U}) \boldsymbol{q}) = \boldsymbol{S}(\boldsymbol{U}), \qquad \boldsymbol{q} + \nabla_\perp \boldsymbol{U} = 0$$

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Solution with HDG approach

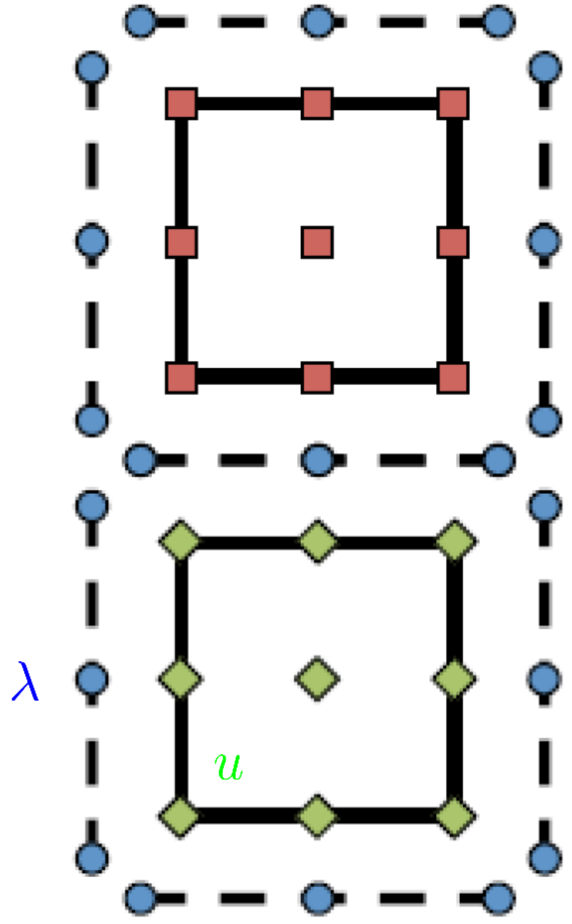
- Implementation: models & operators
  - Model: build matrix contributions at the element level
  - Operator: construct contributions from individual terms in the model
  - Flexibility to implement different models and discretization options

$$\begin{bmatrix} A_{uu} & A_{uq} & A_{ul} \\ A_{qu} & A_{qq} & A_{ql} \\ A_{lu} & A_{lq} & A_{ll} \end{bmatrix} \begin{bmatrix} u \\ q \\ \lambda \end{bmatrix} = \begin{bmatrix} B_u \\ B_q \\ B_l \end{bmatrix}$$

model eq.

$$q + \nabla u = 0$$

trace eq.



Breakdown into operators

- Diffusion operator: implicit

$$\boldsymbol{K}(\boldsymbol{U})\boldsymbol{q} = \begin{bmatrix} \kappa_{1,1}(\boldsymbol{U}) & \kappa_{1,2}(\boldsymbol{U}) & \dots & \kappa_{1,\mathrm{nEq}}(\boldsymbol{U}) \\ \dots & \dots & \dots & \dots \\ \kappa_{\mathrm{nEq},1}(\boldsymbol{U}) & \kappa_{\mathrm{nEq},2}(\boldsymbol{U}) & \dots & \kappa_{\mathrm{nEq},\mathrm{nEq}}(\boldsymbol{U}) \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1 \\ \dots \\ \boldsymbol{q}_{\mathrm{nEq}} \end{bmatrix}$$

- For the 2D interchange model, with $\boldsymbol{U} = \{\boldsymbol{U}_1, ..., \boldsymbol{U}_5\}^T = \{n, n_\mathrm{n}, p_\mathrm{e}, p_\mathrm{i}, W\}^T$

$$\boldsymbol{K}(\boldsymbol{U}) = \begin{bmatrix} D_\perp & 0 & 0 & 0 & 0 \\ 0 & \frac{\boldsymbol{U}_4}{\boldsymbol{U}_1(K_{cx}+K_i)} & 0 & \frac{\boldsymbol{U}_2}{\boldsymbol{U}_1(K_{cx}+K_i)} & 0 \\ \frac{5}{2}\boldsymbol{U}_3 D_\perp & 0 & \chi_{\perp,e}\boldsymbol{U}_1 & 0 & 0 \\ \frac{5}{2}\boldsymbol{U}_4 D_\perp & 0 & 0 & \chi_{\perp,i}\boldsymbol{U}_1 & 0 \\ 0 & 0 & 0 & 0 & \nu \end{bmatrix}$$

$\Rightarrow$ time lagging for the diffusion matrix: evaluate at $\boldsymbol{U}^n$

Breakdown into operators

- Advection operator: explicit, upwind

$$\boldsymbol{C}(\boldsymbol{U}) = \begin{bmatrix} \boldsymbol{U}_1 \mathbf{v}_1 \\ \dots \\ \boldsymbol{U}_{\mathrm{nEq}} \mathbf{v}_{\mathrm{nEq}} \end{bmatrix}$$

- Reaction (implicit, time-lagging) and source (explicit) operators

$$\boldsymbol{S}(\boldsymbol{U}) = \begin{bmatrix} R_{1,1}(\boldsymbol{U}) & R_{1,2}(\boldsymbol{U}) & \dots & R_{1,\mathrm{nEq}}(\boldsymbol{U}) \\ \dots & \dots & \dots & \dots \\ R_{\mathrm{nEq},1}(\boldsymbol{U}) & R_{\mathrm{nEq},2}(\boldsymbol{U}) & \dots & R_{\mathrm{nEq},\mathrm{nEq}}(\boldsymbol{U}) \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_1 \\ \dots \\ \boldsymbol{U}_{\mathrm{nEq}} \end{bmatrix}$$

$$\boldsymbol{S}(\boldsymbol{U}) = \begin{bmatrix} f_1(\boldsymbol{U}) \\ \dots \\ f_{\mathrm{nEq}}(\boldsymbol{U}) \end{bmatrix}$$

Simple advection-diffusion-reaction model

- Single advection-diffusion-reaction equation with linear analytical solution $\bar{u} = 0.1x + 0.2y + 0.3$

$$\partial_t u + \boldsymbol{\nabla} \cdot (u\boldsymbol{v} - D\nabla u) = S - Ru \quad \text{on} \quad \Omega$$
$$u = \bar{u}|_{\partial\Omega} \quad \text{on} \quad \partial\Omega$$

Solenoidal velocity field $\boldsymbol{v} = \{-0.2, 0.1\}^T$; constant, isotropic diffusion tensor D

Source chosen to have constant R and $S = R\bar{u}$

- *N* coupled equations, linear analytical solutions $\bar{u}_i = i\bar{u}$

$$\partial_t u_i + \boldsymbol{\nabla} \cdot \left(u_i\boldsymbol{v} - \sum_{j=1,\mathrm{N}} D_{i,j} \nabla u_j \right) = S_i - \sum_{j=1,\mathrm{N}} R_{i,j} u_j \quad \text{on} \quad \Omega$$
$$u_i = \bar{u}_i|_{\partial\Omega} \quad \text{on} \quad \partial\Omega$$

Same velocity field; isotropic diffusion tensor with $D_{i,j} = \frac{i}{jN} D$; source rate $R_{i,j} = \frac{i}{jN} R$

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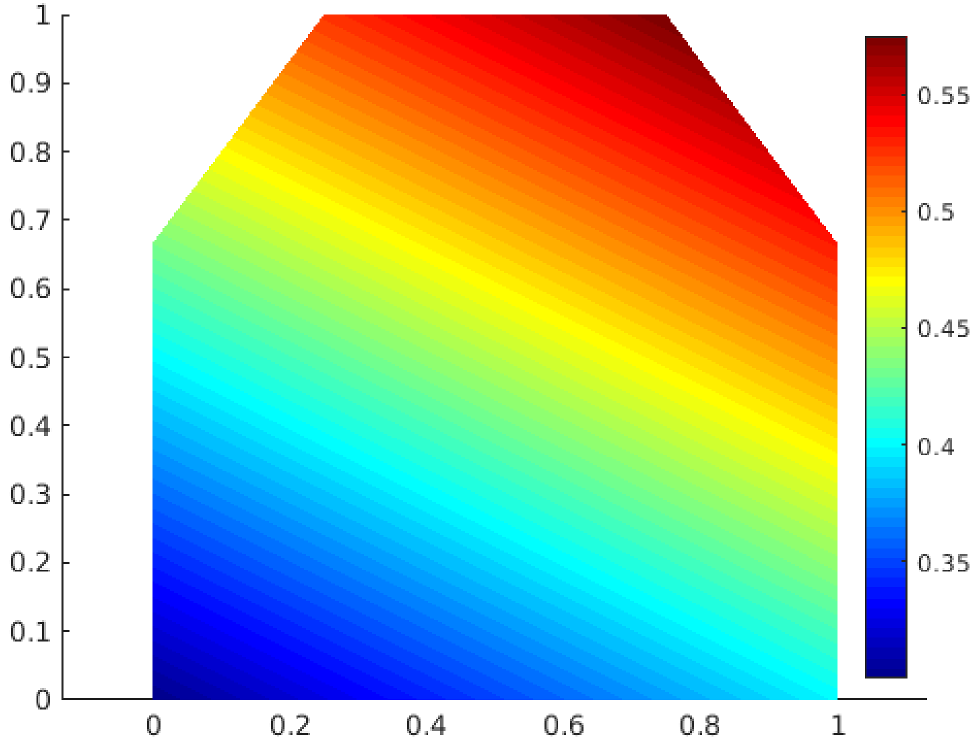
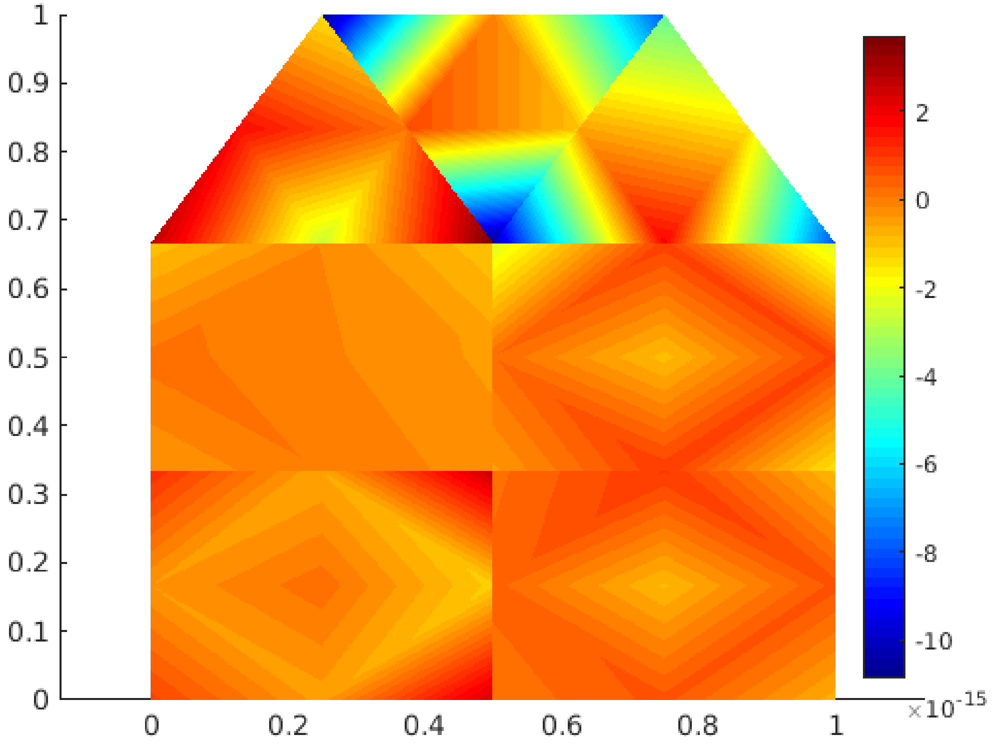
Simple advection-diffusion-reaction model

Solution (-)



Error (-)



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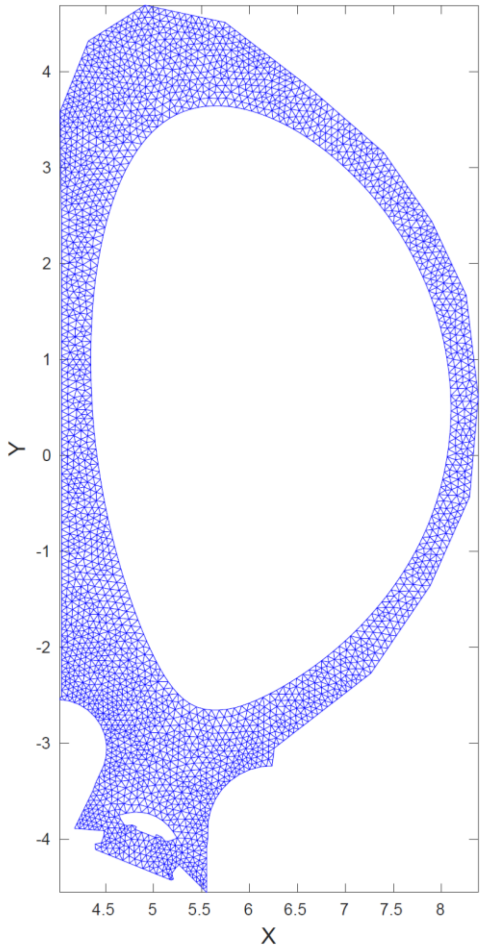
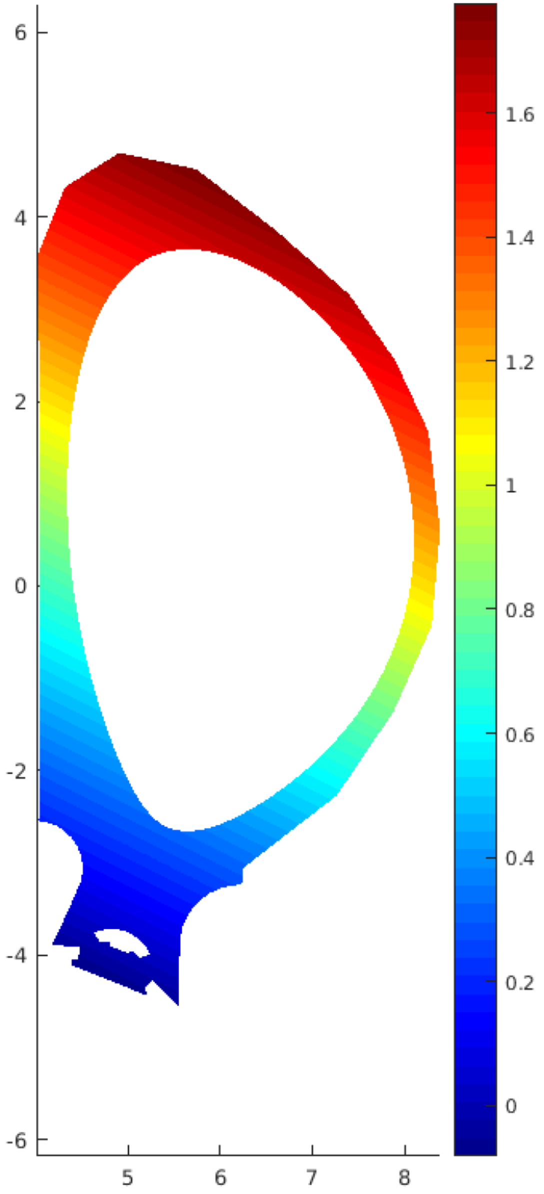
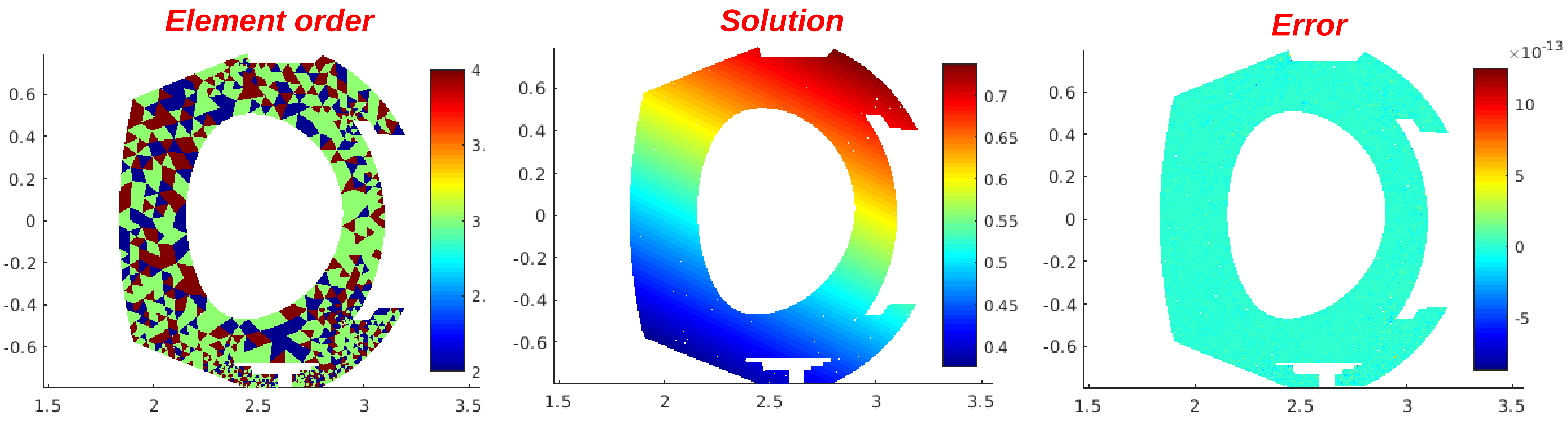
Solution in tokamak geometry





Diffusion equation, arbitrary element order



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Time (in)dependent equations

- Additional "diffusion-reaction" equation for potential, time-independent

$$\partial_t n + \nabla \cdot \left(n(\boldsymbol{u}_E - T_e \boldsymbol{K}) - D_\perp \nabla_\perp n \right) = S_n$$

$$\partial_t n_n + \nabla \cdot \left(\frac{n_{n,eq}}{n} (n\boldsymbol{u}_E + p_i \boldsymbol{K}) - \frac{p_i}{B} \boldsymbol{b} \times \nabla_\perp \left(\frac{n_{n,eq}}{n} \right) - \frac{1}{n(K_{cx} + K_i)} \nabla_\perp (T_i n_n) \right) = S_{n_n}$$

$$\partial_t p_e + \frac{5}{3} \nabla \cdot \left(p_e \boldsymbol{u}_E - \frac{p_e^2}{n} \boldsymbol{K} - \frac{p_e}{n} D_\perp \nabla_\perp n \right) - \frac{2}{3} \nabla \cdot (\chi_{\perp,e} n \nabla_\perp T_e) = S_{p_e}$$

$$\partial_t p_i + \frac{5}{3} \nabla \cdot \left(p_i \boldsymbol{u}_E + \frac{p_i^2}{n} \boldsymbol{K} - \frac{p_i}{n} D_\perp \nabla_\perp n \right) - \frac{2}{3} \nabla \cdot (\chi_{\perp,i} n \nabla_\perp T_i) = S_{p_i}$$

$$\partial_t W + \nabla \cdot \left(W (\boldsymbol{u}_E + T_i \boldsymbol{K}) - \nu \nabla_\perp W \right) + \nabla \cdot ((p_i + p_e + p_n) \boldsymbol{K}) = S_W$$

$$-\nabla \cdot \left(\frac{\nabla_\perp \phi}{B^2} + \frac{\nabla_\perp p_i}{B^2} \right) = -W$$

$\Rightarrow$ set of coupled equations for $\boldsymbol{U} = \{\boldsymbol{U}_1, ..., \boldsymbol{U}_5, \boldsymbol{U}_6\}^T = \{n, n_\mathrm{n}, p_\mathrm{e}, p_\mathrm{i}, W, \phi\}^T$

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Next steps

- Implement toy model

$$\partial_t n + \nabla \cdot (n\boldsymbol{u}_E - D_\perp \nabla_\perp n) = 0$$
$$n - \nabla \cdot \left(\frac{\nabla_\perp \phi}{B} \right) = 0$$

- Implement full 2D interchange model