# Averaging turbulence data for the development of mean-field turbulent transport closure models

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# 1 Objective

The objective of this document is to provide a concise overview of averaged turbulence code data which is expected to be helpful for the development of turbulence closure models to be used in mean-field transport codes.

The turbulence code data, which is expected to have a sufficiently fine spatial and temporal resolution to capture the fine details of the flow fields, lies at the basis for developing and calibrating the self-consistent turbulence closure model. Namely by combining and averaging particular quantities of this turbulence data, mean-field reference data will be obtained which will aid with the aforementioned 2 tasks.

First, section 2 introduces what is meant with averaging and mean-field data. Then, section 3 lists a number of averaged quantities which would be of particular interest in developing mean-field models.

The averaging methodology, the quantities of interest and the transport equations presented in this document are largely based on Ref. [7]. Some of the text in this document has also been copied from this source.

# 2 Definitions and calculation of averages

## 2.1 Averaging operators

In line with the Reynolds Averaging approach commonly used in hydrodynamic turbulence modelling, each turbulent quantity $u$ (which varies chaotically in space and time) is decomposed in an ensemble average component $\bar{u}$ and a fluctuating component $u'$ according to [13]

$$u = \bar{u} + u', \tag{1}$$

$$\bar{u} \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} u^{(i)}. \tag{2}$$

The latter formula defines the ensemble average where $u^{(i)}$ is an individual realization of the flow. In this text, we assume the turbulent flows to evolve towards an ergodic state, meaning that a (long time) statistical steady state of the flow exists and that the time average of it converges to the ensemble average:

$$\bar{u} \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} u^{(i)} = \lim_{T\to\infty} \frac{1}{T} \int_0^T u dt. \tag{3}$$

In addition to the Reynolds average, the Favre or density weighted average will also be used. This is defined as follows [6]:

$$u = \tilde{u} + u'', \tag{4}$$

$$\tilde{u} \triangleq \frac{\overline{nu}}{\bar{n}}. \tag{5}$$

This Favre average is particularly useful when transport equations with variable density need to be averaged, as it allows to limit the number of closure terms in that case. The Reynolds and Favre decompositions imply the following relationships:

$$\overline{u'} = 0, \qquad \overline{nu''} = 0, \tag{6}$$

$$\overline{nu} = \bar{n}\bar{u} + \overline{n'u'} = \bar{n}\tilde{u}, \qquad \overline{nu_1u_2} = \bar{n}\tilde{u}_1\tilde{u}_2 + \overline{nu_1''u_2''}. \tag{7}$$

Note also that the averaging operator $\bar{\ }$ commutes with time and space derivatives, but the Favre operator $\tilde{\ }$ does not:

$$\overline{\nabla u} = \nabla\bar{u}, \qquad \widetilde{\nabla u} = \nabla\tilde{u} + \frac{\overline{n\nabla(u'')}}{\bar{n}} = \nabla\tilde{u} - \frac{\overline{u''\nabla n}}{\bar{n}} \tag{8}$$

## 2.2 Calculating averages from turbulence code data

To approximate/overcome the theoretical infinite time average in equation 3, an average is instead to be taken over a finite set of $N_t$ time samples $t_i$, i.e.

$$\bar{u} = \lim_{T\to\infty} \frac{1}{T} \int_0^T u dt \approx \frac{1}{N} \sum_{i=1}^{N_t} u(t_i). \tag{9}$$

If the turbulence code simulations features a symmetry plane (as the midplane in slab-cases) or direction $\lambda$ (e.g. the toroidal direction in toroidal geometry or the poloidal/diamagnetic direction in slab cases), average quantities $\bar{u}$ are expected to be uniform in this direction. In order to get more data points for the averaging, which will result in smaller statistical noise on the approximated ensemble averages, the data will additionally be averaged over this symmetry direction (or plane), i.e.

$$\bar{u} \approx \frac{1}{N_t N_\lambda} \sum_{i=1}^{N_t} \sum_{j=1}^{N_\lambda} u(t_i, \lambda_j). \tag{10}$$

Note that this also reduces the size of the average data to be saved, since a spatial direction is effectively removed (next to the time coordinate). This averaging over the symmetry direction is not a necessary part of the methodology presented here though, but is recommended.

The averaging in equations 9 or 10 can either be applied by loading all data points at once, by means of a running average or by running a 'batched' running average. The former approach is rather straightforward. It needs all the data to be available at once, which may require a lot of memory and requires storing a large data set (all variations in space and time need to be saved).

Alternatively, a running average can be used in which the average over all the data is gradually built-up by loading the data time step by time step. Moreover, this running averaging can be calculated during runtime of the turbulence code. For numerical considerations this running average calculation can also be divided into batches to obtain batch running averages. These averages may then also be used to explore the uncertainty on the obtained ensemble-averages (and possibly even the correlations between the different ensemble averages). It is also possible to apply both of these algorithms to the stored turbulence code data though. A schematic of a possible implementation is shown below.

- $\bar{u} = 0$
- For $t_i = 1 : N_t$ (Note: assumption that plasma at $t_i$ in stat. steady state.)
  - run time step to obtain $u(t_i, \lambda_j)$ / load a new time step $u(t_i, \lambda_j)$
  - calc average along symm dir: $\langle u(t_i) \rangle_\lambda = \frac{1}{N_\lambda} \sum_{j=1}^{N_\lambda} u(t_i, \lambda_j)$
  - calc run avg: $\bar{u} = \bar{u} + (\langle u(t_i) \rangle_\lambda - \bar{u})/t_i$
- end for

Hence, in this case very little data storage and memory is needed to calculate the average over the entire time horizon. If the running average is calculated during runtime of the turbulence code, the CPU time of the code will of course increase. This may reduced by not updating the running average at every time step of the turbulence code but only after a (fixed) number of time steps. Note that this isn't a problem as the fields will be strongly correlated in 2 subsequent time steps, thereby providing not much information about the ensemble average of the statistical steady state. Another possible issue with calculating the running average during runtime is to determine when to start building up the running average, i.e. to know from what point on steady state is reached. Determining this will also cost additional CPU-time. Lastly, it needs to be noted that some quantities need to be calculated in a in a post processing step, for instance right after the timestepping-loop, which should be very cheap though.

For instance, Favre averages $\tilde{u}$ need to be calculated in such a post processing step. During the main running averaging step, the (Reynolds) averages $\bar{n}$ and $\overline{nu}$ are calculated. Next, the Favre average itself is calculated in the post processing step as $\tilde{u} = \overline{nu}/\bar{n}$. Quantities involving turbulent fluctuations likewise need to be calculated in such a post processing step. For this, equation 7 or similar relations can be used. As an example, the turbulent kinetic energy (see definition 11) will for example be calculated as $k_E = E_{\mathrm{E}} - \frac{m\tilde{\mathbf{V}}_E^2}{2}$, where $E_{\mathrm{E}} = mn\overline{\mathbf{V}_E^2}/2\bar{n}$. Hence $m\overline{n\mathbf{V}_E^2}/2, \bar{n}$ and $\overline{n\mathbf{V}_E}$ are calculated during the running averaging phase and $E_{\mathrm{E}}$, $\tilde{\mathbf{V}}_E$ and $k_E$ in a next post processing step.

## 2.3 Note about the averaged equations presented

In the remainder of this document, only **low** $\beta$ **plasmas** will be considered. Hence, it is assumed that strong, constant in time, magnetic fields are externally applied and that fluctuations of the magnetic field can be neglected. This means that the magnetic fields can be brought out of the averaging operators. However we will make one exception to this as we allowed for the influence of $A_{||}$ on the parallel momentum equations, which hence will appear in the energy equations.)

## 2.4 Note about the calculation of the averages: time-lagging

Note that when correlations between fluctuations are calculated, the fluctuations have to be evaluated at the same time instant. E.g., if the code updates W,N,Pe,Pi and stores them every Y-timesteps, we won't be able to exactly evaluate the averages if PHI is still from the start of the timestep. Hence in our version of TOKAM2D we first determine the potential at the start of every iteration and then store the (plasma) fields which need to be saved in a dummy-array. When the saving is then requested, we write out the values of that dummy-array.

# 3 Mean-field quantities of interest

The idea of the mean-field closures for the perpendicular turbulent fluxes presented in Refs. [5, 2, 1, 3, 4, 8, 10, 9, 11, 12] is that the latter can be modeled by relating them to quantifiers of the turbulence (such as the turbulent kinetic energy and the turbulent enstrophy) and to other mean-field quantities. In order to further verify this ansatz and to elaborate the closure models, more reference data from more extensive cases is required. Data is required from the turbulent fluxes to be modeled, of the turbulent quantities intended to explain these fluxes [1], and of the regular mean-field quantities to which the turbulent quantities are in turn to be related.

The next three sections define a number of quantities which are of particular interest. Section 3.1 presents a minimum set of quantities of interest (QoIs) to validate the mean-field closure models. Section 3.2 extends this set of QoIs to include more information which then will allow for a more elaborate analysis. Section 3.3 continues to further extend the set, which should allow to further develop the turbulence closure models for complex cases.

## 3.1 Minimum set of QoIs

In order to make sense of what is going on in terms of transport in a simulation, it is needed to have an idea of the simulation setup in terms of magnetic and vessel geometry and the boundary conditions. Additionally information about the used grid (i.e. are the grid cells placed symmetrically- or are they field aligned), the interpretation of the saved fields/fluxes (i.e. do they represent the field at the cell center, at the faces,... are the fluxes located in the cell center or at the faces,...) and the discretization/interpolation schemes used to obtain them. The magnetic geometry is also needed to evaluate the mean-field drifts.

Furthermore, the basic mean-field quantities, which are solved for in mean-field transport codes, such as SOLPS-ITER are required to get a view on the conditions in which the turbulence develops and to model the closure terms. This should include the Reynolds averaged density $\bar{n}$ and electrostatic potential $\bar{\phi}$, the Favre averaged temperatures $\tilde{T}_i$ and $\tilde{T}_e$ (or the Reynolds average pressures, as $\bar{p} = \bar{n}\tilde{T}$), and the Favre averaged parallel velocity $\tilde{\mathbf{V}}_{||}$ (or the Reynolds averaged parallel particle flux $\mathbf{\Gamma}_{n,||} = \overline{n\mathbf{V}_{||}} = \bar{n}\tilde{\mathbf{V}}_{||}$). In addition, their gradients in the plane perpendicular to the magnetic field direction ($\mathbf{b}$) are required to calculate the mean-field ExB and diamagnetic drifts.

Next, the ExB perpendicular fluxes which are assumed to be the dominant transport effect of the turbulent fluctuations are required. These include the average turbulent ExB particle flux $\overline{n'\mathbf{V}'_E}$, ion pressure flux $\overline{nT''_i\mathbf{V}''_E}$, electron pressure flux $\overline{nT''_e\mathbf{V}''_E}$ and parallel momentum flux $\overline{nV''_{||}\mathbf{V}''_E}$.

Note that according to equations 7 the total averaged fluxes can be decomposed into a component involving the mean-field fluxes based on $\bar{\mathbf{V}}$ or $\tilde{\mathbf{V}}$ and

[1] Note that $\nabla n$ for instance can be classified as belonging to this category, as it is used for testing the diffusive ansatz

the common mean-field quantities defined above, and a component due to the turbulent fluxes[2]. The easiest way to calculate the turbulent fluxes is to calculate the total flux (e.g. $\overline{n\mathbf{V}_E}$) and the mean-field flux first (e.g. $\bar{n}\bar{\mathbf{V}}_E$, $\bar{\mathbf{V}}_E$ following from $\bar{\phi}$), and to calculate the turbulent flux as the difference of the two (e.g. $\overline{n'\mathbf{V}'_E} = \overline{n\mathbf{V}_E} - \bar{n}\bar{\mathbf{V}}_E$). Care needs to be taken regarding consistency in discretization when computing such differences.

Lastly, the density weighed turbulent kinetic energy $k_E$ and enstrophy $\zeta_E$ due to the turbulent fluctuations in the ExB-velocity are required. These quantities are defined as

$$\bar{n}k_E \triangleq \frac{m\overline{n\mathbf{V}''^2_E}}{2} \quad \text{and} \quad \bar{n}\zeta_E \triangleq \frac{m\overline{nW''^2_E}}{2}, \tag{11}$$

with $m$ the ion mass, and the ExB velocity and pseudo-vorticity defined as

$$\mathbf{V}_E = \frac{\mathbf{b} \times \nabla\phi}{B} \quad \text{and} \quad W_E \triangleq \nabla \cdot \frac{\nabla_\perp \phi}{B^2}. \tag{12}$$

Again, it is easiest to calculate $k_E$ ($\zeta_E$) as the difference between the total kinetic energy (enstrophy) and the mean-field kinetic energy (enstrophy), see section 2.2. In Refs. [5, 2, 1, 3, 4, 8, 10, 9, 11, 12], $k_E$ and $\zeta_E$ are used to model the turbulent transport coefficients for particles ($D$), pressure ($\chi_{i/e}$), and parallel momentum ($\chi_{m,||}$). These transport coefficients were defined through the following diffusive relations. (Detail: in 3D-turbulence we don't have always transport down the exact gradient direction: $\overline{n'\mathbf{V}'_\perp} \times \nabla_\perp \bar{n} = 0$ )

$$\overline{n'\mathbf{V}'_\perp} = -D\nabla_\perp \bar{n}, \tag{13}$$

$$\overline{nT''_{i/e}\mathbf{V}''_\perp} = -\bar{n}\chi_{i/e}\nabla_\perp \tilde{T}_{i/e}, \tag{14}$$

$$m\overline{nV''_{||}\mathbf{V}''_\perp} = -\chi_{m,||}\bar{n}\nabla_\perp \tilde{V}_{||}. \tag{15}$$

The minimum QoIs introduced in this section are summarised in table 1. These quantities would already allow to check some basic hypotheses. In particular, it could be checked if the relation between $k_E$, $\zeta_E$ and the transport coefficients holds in more general cases.

### 3.2 Basic set of QoIs

The basic set of QoIs includes:

- more mean-field quantities to better understand the mean-field equilibrium and more additional quantities (i.e. $k_{||}$) which may be related to the turbulence by using them in the closure expressions.

[2] Note that $\tilde{\mathbf{V}}$ already accounts for the convection of the mean-fields by the turbulent particle flux.

Table 1: Summary of minimum QoIs.

| **Minimum QoI's** |
| --- |
| Description of the geometry and grid (for instance: R,Z,$\phi$ values of the cell centers for toroidal systems,...) |
| Description of numerical schemes and equations solved, including boundary conditions |
| Description of relevant input parameters |
| basic mean-field quantities $\bar{n}$, $\bar{\phi}$, $\tilde{T}_i$, $\tilde{T}_e$, $\tilde{\mathbf{V}}_{\|\|}$ |
| Precise identification of representation reported (plasma) quantities and fluxes. I.e. : |
| $\bar{n}$ at cell center |
| $n\bar{V}_{E,rad}$ at right face |
| $n\bar{V}_{E,pol}$ at bottom face |
| gradients of mean-field quantities perpendicular to $\mathbf{B}$ (both radial and poloidal/diamagnetic component) |
| main turbulent fluxes: |
| $\overline{n'\mathbf{V}'_E}$ via $\overline{n\mathbf{V}_E} - \bar{n}\overline{\mathbf{V}_E}$ |
| $\overline{nT_i''\mathbf{V}_E''}$, $\overline{nT_e''\mathbf{V}_E''}$, $\overline{nV_{\|\|}''\mathbf{V}_E''}$ via $\overline{nX''\mathbf{V}_E''} = \overline{nX\mathbf{V}_E}$ - $\bar{n}\tilde{V}_E\tilde{X}$ |
| main turbulence identifiers (equation 11): $k_E = \frac{m\overline{n\mathbf{V}_E''^2}}{2\bar{n}}$, $\zeta_E = \frac{m\overline{nW_E''^2}}{2\bar{n}}$ |

- Additional data on the turbulence to include both the ExB and ion diamagnetic turbulent statistics (which will be termed total statistics)[3]
- currently thought dominant terms in the transport equations for $k_\perp$ and $\zeta_\perp$
- the turbulent/anomalous component of the mean-field parallel fluxes and additional info on the parallel dynamics

These quantities will allow to

- check if the ExB-only characteristics of the turbulence allow better predictions of turbulent transport than the total quantities.
- check if turbulent fluctuations are important for parallel transport in the other conservation equations
- check some terms in the balance of $k_E$ and $\zeta_E$ to see how the turblence is driven and how the drive terms vary in space. If found to be important, further analysis could then try to model these drive and sink terms.

[3]For $k, \zeta$ the subscripts will indicate which velocities are included: cfr pg 81 & 173 of [7]: $k_{E*} = \frac{m\overline{n(\mathbf{V}_E+V_*)''^2}}{2\bar{n}}$, $\zeta_{E*} = \frac{m\overline{n(W_E+W_*)''^2}}{2\bar{n}}$and with $W_{E*} = W_E + W_* = \nabla\cdot(\frac{\nabla_\perp\phi}{B^2} + \frac{\nabla_\perp p_{\rm i}}{en_{\rm e}B^2})$

Table 2: Summary of additional QoIs for the basic set.

| **Basic** |
|---|
| parallel (convective) turbulent fluxes $\overline{nT_i''V_{\|\|}''}$, $\overline{nT_e''V_{\|\|}''}$, $\overline{nV_{\|\|}''V_{\|\|}''}$ |
| parallel (conductive) total fluxes $\overline{q_i}$, $\overline{q_e}$, $\overline{\Pi_{pp}}$ |
| Polarization current $\bar{\mathbf{J}}_p$ (eq. 26) and parallel $\bar{J}_{\|\|}$ current |
| additional turbulence identifiers: $k_{E*}$, $\zeta_{,E*}$, $k_{\|\|}$ |
| terms in $k_E$ (and $k_{E*}$) equation |
| DW term $\overline{\nabla_{\|\|}\phi' \cdot \mathbf{J}_{\|\|}'}$, |
| interchange term $\overline{p'\nabla \cdot \mathbf{V}_E'}$, |
| Reynolds stress term $-\overline{mn\mathbf{V}_C''\mathbf{V}_E''} : \nabla\tilde{\mathbf{V}}_E^T$, |
| + the stresses themselves $\Pi_{RS,E} = \overline{mn\mathbf{V}_E''\mathbf{V}_E''}$ |
| anomalous parallel transport term $\nabla \cdot (\overline{\phi'\mathbf{J}_{\|\|}'})$ |
| + flux $\overline{\phi'\mathbf{J}_{\|\|}'}$ separately |
| terms in $\zeta_E$ (and $\zeta_{,E*}$) equation |
| parallel current term $\overline{W_E''\nabla \cdot \mathbf{J}_{\|\|}}$, |
| + flux $\overline{W_E''\mathbf{J}_{\|\|}}$ separately |
| interchange term $\overline{W_E''\nabla \cdot \mathbf{J}_*}$, |
| viscous dissipations $\overline{W_E''\nabla \cdot \mathbf{J}_{p,\Pi}}$, |
| Reynolds stress-like term $-\overline{nW_E''\mathbf{V}_C''} \cdot \nabla\tilde{W}_E$ |
| + the stresses themselves $\overline{nW_E''\mathbf{V}_C''}$ |

### 3.3 Extensive set of QoIs

In order to obtain a very complete mean-field picture of the plasma (fluid) all terms present in the mean-field equations have to be evaluated, including the source, sink and transport terms of all energy equations.

For this a 'full' evaluation of the mean-field equations has been derived, which includes: kinetic energy equations for both the perpendicular and parallel components of the turbulent and mean-field parts (hence 4 equations); enstrophy equations for the turbulent and mean-field parts and ion and electron thermal energy equations.

Possibly also magnetic energy, continuity, parallel (ion) momentum and Ohm's law to be added? For instance, the parallel momentum would be nice as we need to check the importance of the reynoldsstresses/anomalous parallel momentum flux. Similarly while doing a complete mean-field analysis of the SOL allows to verify that -according then to the 3D codes- that the anomalous perpendicular fluxes are the most important to be closed first, it is likely 'out of scope' to propose closure relations for possible other closure terms popping up in those equations. However, doing the analysis allows to check for the missing gaps in RANS modelling, which is part of our final deliverable. (this then also allows for checking whether the BC's at the target remain the same,...).

Table 3: Summary of extensive QoIs.

| **Extensive** |
|---|
| all terms in the energy transport equations of section A.1 |
| all terms in the enstrophy transport equations of section A.2 |

# A Transport equations

## A.1 Energy equations

In the following kinetic equations (eqn 16 - 19) we take $\mathbf{V}_0$ the ion inertial velocity equal to the $\mathbf{V}_C$ the ion (particle) convection velocity (not making use of gyro-viscous cancellations) and set $\mathbf{V}_0 = \mathbf{V}_C = \mathbf{V}_{||}+\mathbf{V}_E+\mathbf{V}_{*,i}$. However, in the thermal energy equations we use convection with the polarsiation velocity $\mathbf{V}_p$, hence for eqn 20 - 21 we use $\mathbf{V}_C = \mathbf{V}_{||}+\mathbf{V}_E+\mathbf{V}_{*,i}+\mathbf{V}_p$ and $\mathbf{V}_0 = \mathbf{V}_{||}+\mathbf{V}_E+\mathbf{V}_{*,i}$.

Note that the mean-field equations belonging to the plasma model of the turbulence code will need to be derived on a code by code basis. Hence if in the turbulence model no convection with $\mathbf{V}_p$ is included for the termal energy, it will also not be present in the mean-field equation for it.

$$\begin{aligned}
&\frac{\partial \bar{n} E_{\mathrm{E},m}}{\partial t} + \nabla\cdot(\bar{n}E_{\mathrm{E},m}\tilde{\mathbf{V}}_C + mn\overline{\mathbf{V}''_C\mathbf{V}''_E}\cdot\tilde{\mathbf{V}}_E + \bar{\Pi}\cdot\tilde{\mathbf{V}}_E + \bar{\phi}\bar{\mathbf{J}} + \bar{p}\bar{\mathbf{V}}_E) \\
&= \bar{\mathbf{J}}_{||}\cdot\nabla_{||}\bar{\phi} + \bar{p}\nabla\cdot\bar{\mathbf{V}}_E + \nabla\bar{\phi}\cdot\bar{\mathbf{J}}_{p,*} + \bar{\Pi}:\nabla\tilde{\mathbf{V}}_E^T + mn\overline{\mathbf{V}''_C\mathbf{V}''_E}:\nabla\tilde{\mathbf{V}}_E^T \\
&\quad -(\frac{\bar{\mathbf{J}}_{p,E}+\bar{\mathbf{J}}_{p,\Pi}}{\bar{n}})\cdot\overline{n'\nabla\phi'} - mn\overline{V_{||}\frac{D\mathbf{b}}{Dt}}\cdot\tilde{\mathbf{V}}_E - E_{\mathrm{E},m}\bar{S}_{n_\mathrm{i}} + \bar{\mathbf{S}}_m\cdot\tilde{\mathbf{V}}_E. \qquad (16)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial \bar{n} k_E}{\partial t} + \nabla\cdot(\bar{n}k_E\tilde{\mathbf{V}}_C + \frac{mn\overline{\mathbf{V}''^2_E\mathbf{V}''_C}}{2} + \overline{\Pi\cdot\mathbf{V}''_E} + \overline{\phi'\mathbf{J}'} + \overline{p'\mathbf{V}'_E}) \\
&= \overline{\nabla_{||}\phi'\cdot\mathbf{J}'_{||}} + \overline{p'\nabla\cdot\mathbf{V}'_E} + \overline{\nabla\phi'\cdot\mathbf{J}'_{p,*}} + \overline{\Pi:\nabla\mathbf{V}''^T_E} \\
&\quad -mn\overline{\mathbf{V}''_C\mathbf{V}''_E}:\nabla\tilde{\mathbf{V}}_E^T + (\frac{\bar{\mathbf{J}}_{p,E}+\bar{\mathbf{J}}_{p,\Pi}}{\bar{n}})\cdot\overline{n'\nabla\phi'} - mn\overline{V_{||}\frac{D\mathbf{b}}{Dt}\cdot\mathbf{V}''_E} \\
&\quad -\frac{m\overline{S_{n_\mathrm{i}}\mathbf{V}''^2_E}}{2} - m\tilde{\mathbf{V}}_E\cdot\overline{\mathbf{V}''_E\cdot S_{n_\mathrm{i}}} + \overline{\mathbf{S}_m\cdot\mathbf{V}''_E}. \qquad (17)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}(\bar{n}E_{\mathrm{k},m,||}) + \nabla\cdot(m\bar{n}\tilde{\mathbf{V}}_C E_{\mathrm{k},m,||} + mn\overline{\mathbf{V}''_C\mathbf{V}''_{||}}\cdot\tilde{\mathbf{V}}_{||} + \overline{\Pi}\cdot\tilde{\mathbf{V}}_{||}) \\
&= \overline{\Pi}:\nabla\tilde{\mathbf{V}}_{||}^T + mn\overline{\mathbf{V}''_C\mathbf{V}''_{||}}:\nabla\tilde{\mathbf{V}}_{||}^T - \tilde{\mathbf{V}}_{||}\cdot\nabla_{||}\bar{p} \\
&\quad +m\tilde{V}_{||}\overline{\frac{D\mathbf{b}}{Dt}\cdot n\mathbf{V}_{0,\perp}} - E_{\mathrm{k},m,||}\bar{S}_{n_\mathrm{i}} + \tilde{\mathbf{V}}_{||}\cdot\bar{\mathbf{S}}_m. \qquad (18)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}(\bar{n}k_{||}) + \nabla\cdot(\bar{n}\tilde{\mathbf{V}}_C k_{||} + \frac{mn\overline{\mathbf{V}''_C\mathbf{V}''^2_{||}}}{2} + \overline{\Pi\cdot\mathbf{V}''_{||}}) \\
&= \overline{\Pi:\nabla\mathbf{V}''^T_{||}} - mn\overline{\mathbf{V}''_C\mathbf{V}''_{||}}:\nabla\tilde{\mathbf{V}}_{||}^T - \overline{\mathbf{V}''_{||}\cdot\nabla_{||}p} \\
&\quad +m\overline{V''_{||}\frac{D\mathbf{b}}{Dt}\cdot n\mathbf{V}_{0,\perp}} - \frac{\overline{S_{n_\mathrm{i}}\mathbf{V}''^2_{||}}}{2} - \tilde{\mathbf{V}}_{||}\cdot\overline{\mathbf{V}''_{||}\cdot S_{n_\mathrm{i}}} + \overline{\mathbf{V}''_{||}\cdot\mathbf{S}_m}. \qquad (19)
\end{aligned}$$

$$
\begin{aligned}
&\frac{3}{2}\frac{\partial \bar{p}_i}{\partial t} + \nabla\cdot(\frac{3}{2}\mathbf{\Gamma}_{p_\mathrm{i},E} + \frac{5}{2}\mathbf{\Gamma}_{p_\mathrm{i},||} + \frac{5}{2}\mathbf{\Gamma}_{p_\mathrm{i},p} + \frac{5}{2}\frac{\mathbf{b}\times\nabla\overline{p_\mathrm{i}T_i}}{eB} + \bar{\mathbf{q}}_{||,i}),\\
&= -\overline{p_\mathrm{i}\nabla\cdot\mathbf{V}_E} + \overline{\mathbf{V}_{||}\cdot\nabla p_\mathrm{i}} + \overline{\mathbf{V}_p\cdot\nabla p_\mathrm{i}} - \overline{\Pi:\nabla\mathbf{V}_0^T} - \bar{Q}_\mathrm{ei} + \frac{3}{2}\bar{S}_{p_\mathrm{i}},
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
&\frac{3}{2}\frac{\partial \bar{p}_e}{\partial t} + \nabla\cdot(\frac{3}{2}\mathbf{\Gamma}_{p_\mathrm{e},E} + \frac{5}{2}\mathbf{\Gamma}_{p_\mathrm{e},||} + \frac{5}{2}\frac{\mathbf{b}\times\nabla\overline{p_\mathrm{e}T_e}}{eB} + \bar{\mathbf{q}}_{||,e}),\\
&= -\overline{p_\mathrm{e}\nabla\cdot\mathbf{V}_E} + \overline{\mathbf{V}_{||}\cdot\nabla p_\mathrm{e}} - \overline{\frac{\mathbf{J}_{||}}{ne}\cdot\nabla p_\mathrm{e}} + \overline{\frac{\mathbf{J}}{en}\cdot\mathbf{R}_{ei}} + \bar{Q}_\mathrm{ei} + \frac{3}{2}\bar{S}_{p_\mathrm{e}}.
\end{aligned} \tag{21}
$$

Furthermore, we define fluxes and polarisation currents as follows.

$$
\mathbf{\Gamma}_{pi/e,E} \triangleq \overline{nT_{i/e}\mathbf{V}_E} = \tilde{T}_{i/e}\mathbf{\Gamma}_{n,E} + \overline{nT''_{i/e}\mathbf{V}''_E} \triangleq \mathbf{\Gamma}_{pi/e,m,E} + \mathbf{\Gamma}_{pi/e,t,E}, \tag{22}
$$

$$
\mathbf{\Gamma}_{p_\mathrm{i},||} \triangleq \overline{nT_i\mathbf{V}_{||}} = \tilde{T}_i\mathbf{\Gamma}_{n,||} + \overline{nT''_i\mathbf{V}''_{||}} \tag{23}
$$

$$
\mathbf{\Gamma}_{p_\mathrm{e},||} \triangleq \overline{nT_e\mathbf{V}_{||}} - \overline{\frac{p_\mathrm{e}\mathbf{J}_{||}}{en}} = \tilde{T}_e\mathbf{\Gamma}_{n,||} + \overline{nT''_e\mathbf{V}''_{||}} - \frac{\tilde{T}_e\bar{\mathbf{J}}_{||}}{e} - \frac{\overline{T''_e\mathbf{J}_{||}}}{e} \tag{24}
$$

$$
\mathbf{\Gamma}_{p_\mathrm{i},p} \triangleq \overline{nT_i\mathbf{V}_p} = \bar{n}\tilde{T}_i\tilde{\mathbf{V}}_p + \overline{nT''_i\mathbf{V}''_p} \tag{25}
$$

$$
\begin{aligned}
\mathbf{J}_p &\triangleq -\left(\frac{\partial n\mathbf{V}_0}{\partial t} + \nabla\cdot(n\mathbf{V}_C\mathbf{V}_0)\right)\times\frac{m\mathbf{b}}{B} - \frac{\nabla\cdot\Pi}{B}\times\mathbf{b} + \frac{\mathbf{S}_m\times\mathbf{b}}{B}\\
&\triangleq -\underbrace{\left(\frac{\partial n\mathbf{V}_{||}}{\partial t} + \nabla\cdot n\mathbf{V}_C\mathbf{V}_{||}\right)\times\frac{m\mathbf{b}}{B}}_{\mathbf{J}_{p,||}} - \underbrace{\left(\frac{\partial n\mathbf{V}_E}{\partial t} + \nabla\cdot n\mathbf{V}_C\mathbf{V}_E\right)\times\frac{m\mathbf{b}}{B}}_{\mathbf{J}_{p,E}}\\
&\quad - \underbrace{\left(\frac{\partial n\mathbf{V}_{*,i}}{\partial t} + \nabla\cdot n\mathbf{V}_C\mathbf{V}_{*,i}\right)\times\frac{m\mathbf{b}}{B}}_{\mathbf{J}_{p,*}} \underbrace{- \frac{\nabla\cdot\Pi}{B}\times\mathbf{b} + \frac{\mathbf{S}_m\times\mathbf{b}}{B}}_{\mathbf{J}_{p,\Pi}}
\end{aligned} \tag{26}
$$

$$
\mathbf{J}_{p,||} \triangleq -\left(\frac{\partial n\mathbf{V}_{||}}{\partial t} + \nabla\cdot n\mathbf{V}_C\mathbf{V}_{||}\right)\times\frac{m\mathbf{b}}{B} = -\frac{D\mathbf{b}}{Dt}\times\frac{mn\mathbf{V}_{||}}{B}. \tag{27}
$$

$$
\begin{aligned}
\mathbf{J}_{p,E} &\triangleq -(\frac{\partial n\mathbf{V}_E}{\partial t} + \nabla\cdot(n\mathbf{V}_C\mathbf{V}_E))\times\frac{m\mathbf{b}}{B}\\
&= -\frac{m}{B}\frac{\partial}{\partial t}n\mathbf{U}_E + \frac{m}{B}\nabla\cdot(n\mathbf{V}_C\mathbf{U}_E) - \frac{D\mathbf{b}}{Dt}\times\frac{mn\mathbf{V}_E}{B}
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
\mathbf{J}_{p,*} &\triangleq -(\frac{\partial n\mathbf{V}_{*,i}}{\partial t} + \nabla\cdot(n\mathbf{V}_C\mathbf{V}_{*,i}))\times\frac{m\mathbf{b}}{B}\\
&= -\frac{m}{B}\frac{\partial}{\partial t}n\mathbf{U}_{*,i} + \frac{m}{B}\nabla\cdot(n\mathbf{V}_C\mathbf{U}_{*,i}) - \frac{D\mathbf{b}}{Dt}\times\frac{mn\mathbf{V}_{*,i}}{B}
\end{aligned} \tag{29}
$$

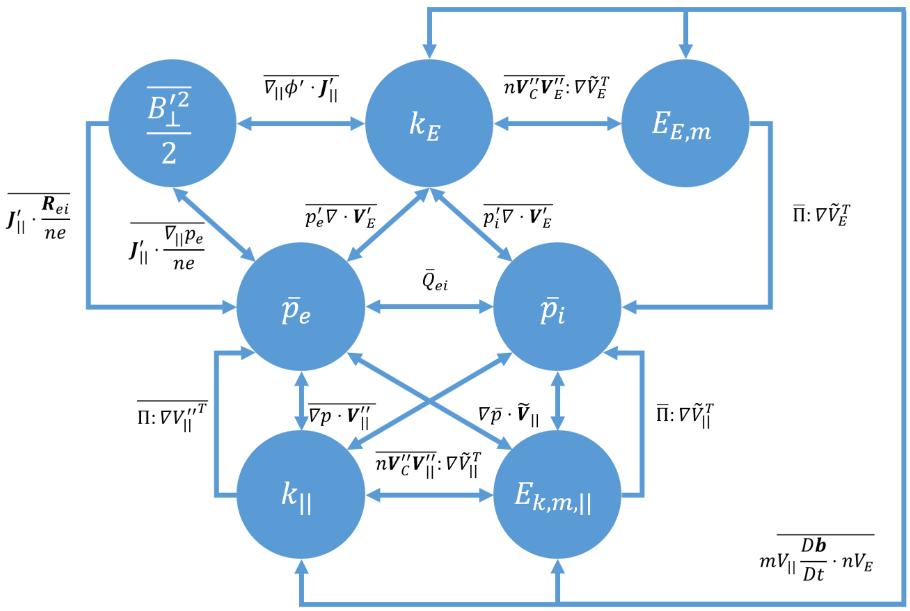


Figure 1: Schematic representation of the main energy transfer channels between the different energy forms in plasma edge turbulence. Adapted from Ref. [14].

$$\mathbf{U}_E \triangleq -\mathbf{b} \times \mathbf{V}_E = \frac{\nabla_\perp \phi}{B}, \qquad \mathbf{U}_{*,i} \triangleq -\mathbf{b} \times \mathbf{V}_{*,i} = \frac{\nabla_\perp p_{\mathrm{i}}}{enB}. \tag{30}$$

## A.2 Enstrophy equations

$$\begin{aligned}\frac{\partial \bar{n}\zeta_{mean,E}}{\partial t} + \nabla \cdot (\bar{n}\zeta_{mean,E}\tilde{\mathbf{V}}_C + \overline{mnW''_E\mathbf{V}''_C} \cdot \tilde{W}_E) = \tilde{W}_E\nabla \cdot \bar{\mathbf{J}}_{||} \\ +\tilde{W}_E\nabla \cdot \bar{\mathbf{J}}_* + \tilde{W}_E\nabla \cdot \bar{\mathbf{J}}_{p,\Pi} + \tilde{W}_E\nabla \cdot \bar{\mathbf{J}}_{p,*} + \tilde{W}_E\nabla \cdot \bar{\mathbf{J}}_{p,||} \\ +\overline{nW''_E\mathbf{V}''_C} \cdot \nabla\tilde{W}_E + \tilde{W}_E\bar{S}_{W_E,cor} - \zeta_{mean,E}\bar{S}_{n_{\mathrm{i}}},\end{aligned} \tag{31}$$

$$\begin{aligned}\frac{\partial \bar{n}\zeta_{turb,E}}{\partial t} + \nabla \cdot (\bar{n}\zeta_{turb,E}\tilde{\mathbf{V}}_C + \frac{\overline{mnW''^2_E\mathbf{V}''_C}}{2}) = \overline{W''_E\nabla \cdot \mathbf{J}_{||}} \\ +\overline{W''_E\nabla \cdot \mathbf{J}_*} + \overline{W''_E\nabla \cdot \mathbf{J}_{p,\Pi}} + \overline{W''_E\nabla \cdot \mathbf{J}_{p,*}} + \overline{W''_E\nabla \cdot \mathbf{J}_{p,||}} \\ -\overline{nW''_E\mathbf{V}''_C} \cdot \nabla\tilde{W}_E + \overline{W''_E S_{W_E,cor}} - \frac{\overline{W''^2_E S_{n_{\mathrm{i}}}}}{2} - \tilde{W}_E\overline{W''_E S_{n_{\mathrm{i}}}}.\end{aligned} \tag{32}$$

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