Constants of motion and nonlinear dynamics*

Fulvio Zonca[1,2]

[1]Center for Nonlinear Plasma Science and ENEA C.R. Frascati, 00044 Frascati, Italy
[2]IFTS, School of Physics, Zhejiang University, Hangzhou, 310027 P.R. China

*Acknowledgement: Liu Chen, Zhiyong Qiu, Matteo V. Falessi

Frascati, January 27th, 2023

General concepts

The extended phase space

- The Least Action Principle, underlying the derivation of the equations of motion, can be rewritten introducing a parameter ζ, with respect to which all variations are independent; i.e.

$$\delta\left[\int\left(\boldsymbol{p}\frac{d\boldsymbol{q}}{d\zeta}-H(\boldsymbol{p},\boldsymbol{q},t)\frac{dt}{d\zeta}\right)d\zeta\right]=0$$

- From the form of the variational principle, it is intuitive that we can define an extended phase space setting in $2N+2$ dimensions by letting $\bar{p}_i=p_i$, $\bar{q}_i=q_i$ for $i=1,N$ and $\bar{p}_{N+1}=-H$, $\bar{q}_{N+1}=t$.

A.J. Lichtenberg and M.A. Lieberman, *Regular and Chaotic Dynamics*, Second Edition, Springer (2010)

General concepts

The reduced phase space

- Given the concept of the extended phase space, every system can be described as autonomous system, i.e. with an Hamiltonian that does not explicitly depend on time; i.e.

$$H(\boldsymbol{p}, \boldsymbol{q}) = H_0$$

- Conversely, one could use this equation to solve one of the momenta, say p_N, as a function of $\bar{p}_i = p_i$ and $\bar{q}_i = q_i$ for $i = 1, N-1$ and $\zeta = q_N$ considered as a parameter; $p_N = p_N(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, q_N)$. Letting $\bar{H} = -p_N(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, q_N)$, the equations of motion in the $2N-2$ dimensional reduced phase space are

$$\frac{d\bar{p}_i}{d\zeta} = -\frac{\partial \bar{H}}{\partial \bar{q}_i} = -\frac{1}{\dot{q}_N}\frac{\partial H}{\partial q_i} \quad ; \quad \frac{d\bar{q}_i}{d\zeta} = \frac{\partial \bar{H}}{\partial \bar{p}_i} = \frac{1}{\dot{q}_N}\frac{\partial H}{\partial p_i}$$

General concepts

- A very useful application of the notion of reduced phase space is the definition of a Poincaré surface of section.
- For a 2D system, the motion is bound to occur within the energy surface $H(p_1, p_2, q_1, q_2) = H_0$, which can be written as
- If the motion is bounded in the phase space, the motion can repeatedly cross the plane $q_2 =$ const, which is a convenient choice of the surface of section, coinciding with the reduced phase space of the original Hamiltonian system.
- In general, the subsequent crossings of the motion in the (p_1, q_1) surface of section can occur everywhere. However, if a constant of motion exist, in addition to H_0, then $I(p_1, p_2, q_1, q_2) =$ const in addition to $p_2 = p_2(p_1, q_1, q_2)$. Therefore

$$p_1 = p_1(q_1, q_2)$$

  and the subsequent crossings of the motion in the (p_1, q_1) surface of section must lie on one curve.
- Vice-versa, the fact that the motion in the (p_1, q_1) surface of section lies on a curve can be used as evidence of the existence of a constant of motion in addition to H_0.

General concepts

- Higher dimensionality:
  - Nearly integrable periodic systems allow to generally look into surface of section plots
  - Hamiltonian mapping
  - Close to resonance [Chirikov 79] «universal description of a nonlinear resonance» ➔ 1D NL pendulum

- Loss of periodicity:
  - Use of Finite Time Lyapunov Exponents (FTLE) [Falessi et al JPP 2015]
  - Hamiltonian Mapping/Kinetic Poincaré plots [RB White CNSNC 2012; Briguglio et al PoP 2014; Zonca et al NJP & PPCF 2015]

General concepts

- Gyrokinetic particle dynamics:
  - System with 2 degrees of freedom (μ invariance)
    ➔ Arnold's diffusion (not this talk)
  - Loss of periodicity may hide resonance structures
  - Reduction to 1 degree of freedom if another constant of motion can be identified (Hamiltonian in extended phase space) [Zonca et al NJP15]

$$K = H_0 + e\langle\delta\psi\rangle - \bar{H}$$

$$\Pi_\phi \equiv P_\phi + (e/c)(F/B_0)\left\langle\delta A_\parallel\right\rangle$$

$$\dot{\bar{H}} = \dot{H}_0 + \left(\frac{\partial}{\partial t} + \dot{\mathbf{X}}\cdot\boldsymbol{\nabla}\right) e\left\langle\delta\phi - \frac{v_\parallel}{c}\delta A_\parallel\right\rangle = e\frac{\partial}{\partial t}\left(\langle\delta\phi\rangle - \frac{v_\parallel}{c}\left\langle\delta A_\parallel\right\rangle\right)$$

$$\dot{\Pi}_\phi \simeq -e\left(\frac{\partial}{\partial\xi}\langle\delta\phi\rangle - \frac{v_\parallel}{c}\frac{\partial}{\partial\xi}\left\langle\delta A_\parallel\right\rangle\right)$$

Nonlinear invariant of motion

- Gyrokinetic particle dynamics: existence of general invariant of motion connected with conservation of Hamiltonian in extended phase space

$$\frac{\partial}{\partial t}\dot{\Pi}_\phi + \frac{\partial}{\partial \xi}\dot{\bar{H}} = 0 \quad ,$$

which readily yields $\Pi_\phi - (n/\omega)\bar{H}$ =const for a single n fixed ω mode.

- ➢ What happens for chirping frequency
- ➢ What happens for multi-mode
- ➢ All this is discussed in NJP15

$$K = H_0 + e\left\langle \delta\psi \right\rangle - \bar{H}$$

$$\Pi_\phi \equiv P_\phi + (e/c)(F/B_0)\left\langle \delta A_\parallel \right\rangle$$

Nonlinear invariant of motion

- Gyrokinetic particle dynamics: existence of general invariant of motion connected with conservation of Hamiltonian in extended phase space

$$\frac{\partial}{\partial t}\dot{\Pi}_\phi + \frac{\partial}{\partial \xi}\dot{\bar{H}} = 0 \ ,$$

which readily yields $\Pi_\phi - (n/\omega)\bar{H}$ =const for a single n fixed ω mode.

- Single – n chirping mode (wave packet)

$$-i\omega(t)\dot{\Pi}_\phi + in\dot{\bar{H}} = 0$$

- Not enough: we need resonance analysis

$$K = H_0 + e\langle\delta\psi\rangle - \bar{H}$$

$$\Pi_\phi \equiv P_\phi + (e/c)(F/B_0)\langle\delta A_\parallel\rangle$$

Resonance analysis

- Move into the wave-frame and apply method of averaging (basis for perturbation theory)
    - Assume nearly periodic behavior and isolated resonances (primary/linear)
    - Assume system is not dominated by resonances among different degrees of freedom (secondary/nonlinear resonances)
    - Follow NJP15: lifting to the particle phase space/push forward representation into the magnetic drift/banana center frame (effective mode structure decomposition)

$$f(r,\theta,\zeta) = e^{in\zeta} \sum_{m\in\mathbb{Z}} e^{-im\theta} f_{m,n}(r)$$

$$\mapsto f_c(\bar{r}_c,\theta_c,\zeta_c) = e^{in\zeta_c} \sum_{m,\ell\in\mathbb{Z}} \lambda_{m,n}(\bar{r}_c,\theta_c) e^{i\ell\theta_c} \mathscr{F}_{m,n,\ell}(\bar{r}_c) \ ,$$

$$\lambda_{m,n}(\bar{r}_c,\theta_c) = \exp\left[i\left(n\bar{q}(\bar{r}_c) - m\right)\bar{\theta}_c\right]$$

Resonance analysis

$$\mathscr{F}_{m,n,\ell}(\bar{r}_c) = \frac{1}{2\pi}\oint \exp\left\{in\tilde{\Xi}(\theta_c) + i\left[n\bar{q}(\bar{r}_c) - m\right]\tilde{\Theta}_c(\theta_c)\right\} f_{m,n}(\bar{r}_c + \tilde{\rho}(\theta_c))e^{-i\ell\theta_c}d\theta_c \ .$$

$$f(r,\theta,\zeta) \mapsto f_c(\bar{r}_c,\theta_c,\zeta_c) = e^{in\zeta_c}\sum_{m,\ell\in\mathbb{Z}} e^{i\ell\theta_c}\mathscr{P}_{m,n,\ell}\circ f_{m,n}(\bar{r}_c,\theta_c) \ .$$

- n summation implicit
- Resonances: stationary phase(s)
- Near isolated resonance other contributions are suppressed by (method of) averaging

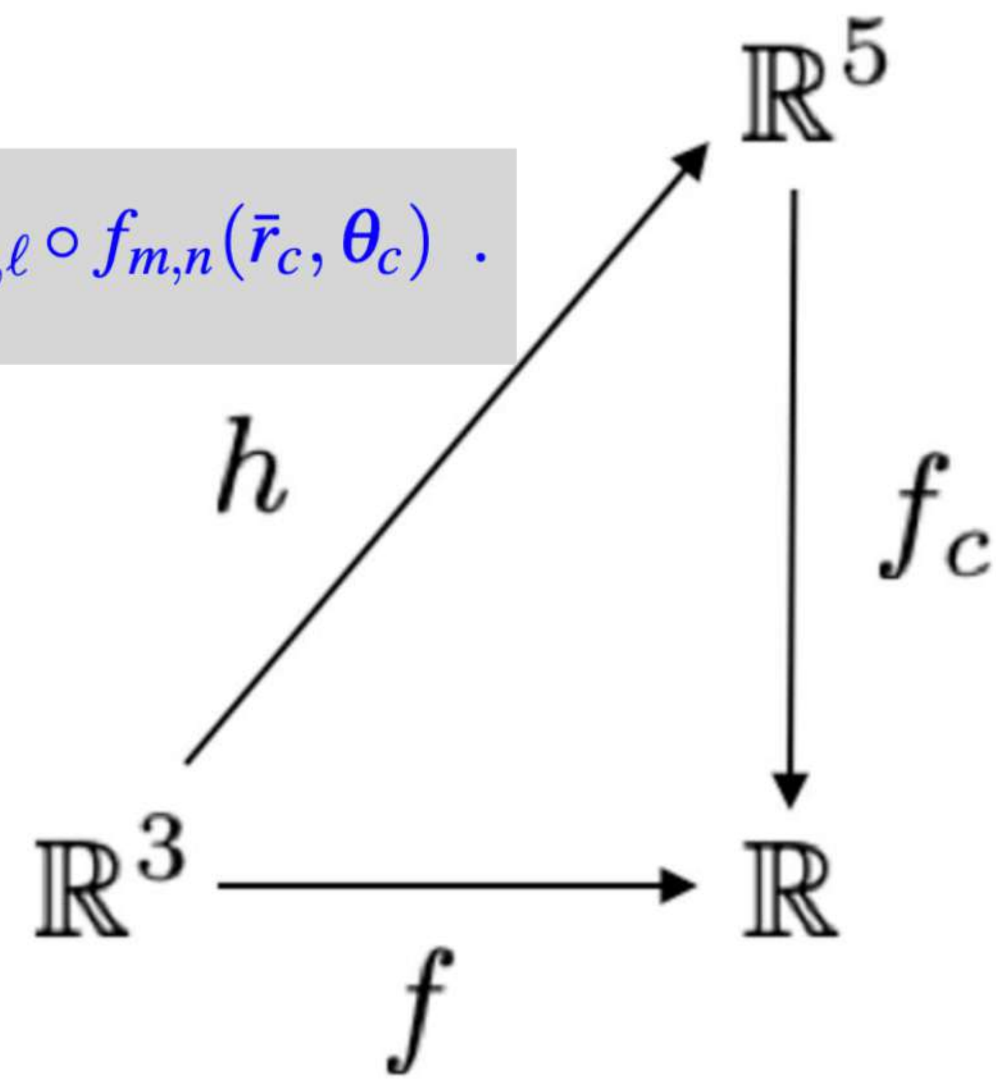

Diagram illustrating the map $h : (r,\theta,\zeta) \mapsto (\bar{r}_c,\theta_c,\zeta_c)$ as lifting of a generic function f to the phase space, such that $f = f_c \circ h$.

Nonlinear invariant of motion

$$\frac{\partial}{\partial t}\dot{\Pi}_\phi + \frac{\partial}{\partial \xi}\dot{\bar{H}} = 0 \quad ,$$

which readily yields $\Pi_\phi - (n/\omega)\bar{H}$ =const for a single n fixed ω mode.

- For isolated resonance

$$-i\omega(t)\mathcal{P}_{m,n,\ell} \circ \dot{\Pi}_{\phi\ m,n} + in\mathcal{P}_{m,n,\ell} \circ \dot{\bar{H}}_{m,n} = 0$$

- Rigorous extension of C constant for single n fixed frequency mode

- Rigorous justification of the statement: phase is the proper canonical conjugate to the conserved extended phase space Hamiltonian (upon application of method of averaging)

Questions/Comments

- NJP15 provides the complete theoretical framework for constructing push-forward/pull-back operators of any fluctuation structure over the action angle space
- This allows a thorough resonance analysis, for isolated/overlapping resonances and for fixed-frequency/chirping modes

- Questions/comments are welcome