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FÜR PLASMAPHYSIK

Implementation of Energy-conserving Hybrid MHD-driftkinetic Model in STRUPHY

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EUROfusion

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Modeling of EP physics

1. **Multi-species fully kinetic code:** `ORB5`, `EUTERPE`, **...**

   Kinetic eqs + Poisson eq + Ampere eq.

2. **Multi-scale hybrid code:** `MEGA`, `HMGC`, `HAGIS`, `STRUPHY`, **...**

   - Hybrid MHD-kinetic simulation

     Kinetic eqs for EPs + MHD eqs.

   - Perturbative hybrid MHD-kinetic simulation

     Kinetic eqs for EPs + LinearMHD eqs.

- **Hybrid MHD-kinetic system:** there are two possible coupling schemes.

Current-coupling:
$$\rho \frac{\partial \mathbf{U}}{\partial t} + \rho(\mathbf{U} \cdot \nabla)\mathbf{U} = (\mathbf{J} - \mathbf{J}_h) \times \mathbf{B} + \rho_h \mathbf{U} \times \mathbf{B} - \nabla p \,.$$

Pressure-coupling:
$$\rho \frac{\partial \mathbf{U}}{\partial t} + \rho(\mathbf{U} \cdot \nabla)\mathbf{U} = \mathbf{J} \times \mathbf{B} - \nabla p - \nabla \cdot \mathbb{P}_h \,.$$

STRUPHY Hybrid fluid-kinetic framework

Goal: "Hybrid fluid-kinetic simulation with a long-time numerical stability."

1. **FEEC-PIC discretization.**

   $\Rightarrow$ no spurious modes from discretization.

   $\Rightarrow$ built-in discrete conservation laws.

2. **Variational (Hamiltonian) Hybrid models:**

   MHD-Vlasov PC and CC

   MHD-Driftkinetic CC

   $\Rightarrow$ Energy-conserving coupling.

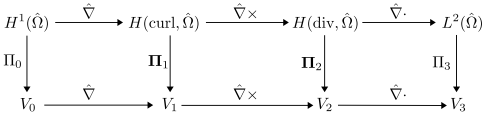
3. **Energy-preserving integrators:**

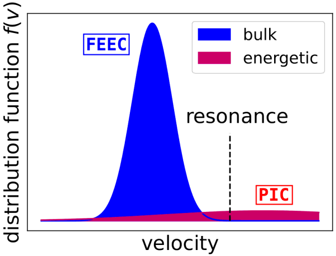
   Hamiltonian splitting.

   Semi-implicit Crank-Nicolson method.

   Integral-preserving integrators.

   $\Rightarrow$ Numerical stability with a large Δt.





Variational (Hamiltonian) Hybrid MHD-driftkinetic Model[1]

MHD $(\rho, \mathbf{U}, \mathbf{B}, p)$:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0\,, \\ \rho \dfrac{\partial \mathbf{U}}{\partial t} + \rho(\mathbf{U} \cdot \nabla)\mathbf{U} = \left(\dfrac{1}{\mu_0} \nabla \times \mathbf{B} + q_h \rho_h \mathbf{U} - \mathbf{J}_h - \nabla \times \mathbf{M}_h \right) \times \mathbf{B} - \nabla p\,, \\ \dfrac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{U}) = 0\,, \\ \dfrac{\partial p}{\partial t} + \nabla \cdot (p\mathbf{U}) + (\gamma - 1) p \nabla \cdot \mathbf{U} = 0\,. \end{cases}$$

Driftkinetic $F(\mathbf{x}, v_\parallel)$:

$$\begin{cases} \dfrac{\partial F}{\partial t} + \dfrac{\partial}{\partial \mathbf{x}} \cdot \left[\dfrac{F}{B_\parallel^*} (v_\parallel \mathbf{B}^* - \mathbf{b}_0 \times \mathbf{E}^*) \right] + \dfrac{q_h}{m_h} \dfrac{\partial}{\partial v_\parallel} \left(\dfrac{F}{B_\parallel^*} \mathbf{B}^* \cdot \mathbf{E}^* \right) = 0\,, \\ \mathbf{B}^* = \mathbf{B} + \dfrac{m_h}{q_h} v_\parallel \nabla \times \mathbf{b}_0\,, \quad \mathbf{E}^* = -\mathbf{U} \times \mathbf{B} - \dfrac{\mu}{q_h} \nabla B_\parallel\,, \quad B_\parallel = \mathbf{b}_0 \cdot \mathbf{B}\,, \quad \mu = \dfrac{m_h v_\perp^2}{2B}\,. \\ \rho_h = \displaystyle\int F \, \mathrm{d}v_\parallel \mathrm{d}\mu\,, \quad \mathbf{J}_h = q_h \int \dfrac{F}{B_\parallel^*} (v_\parallel \mathbf{B}^* - \mathbf{b}_0 \times \mathbf{E}^*) \mathrm{d}v_\parallel \mathrm{d}\mu\,, \quad \mathbf{M}_h = -\int F \mu \mathbf{b}_0 \, \mathrm{d}v_\parallel \mathrm{d}\mu\,. \end{cases}$$

[1] Cesare Tronci (2020) "Variational mean-fluctuation splitting and drift-fluid models"

Energy conservation

- **Conserves the Hamiltonian (total energy of the system):**

$$\mathcal{E}_{\text{total}} = \mathcal{E}_{\text{MHD}} + \mathcal{E}_{\text{EPs}} \,,$$

$$\mathcal{E}_{\text{MHD}} = \underbrace{\frac{1}{2}\int \rho \mathbf{U}^2 \mathrm{d}^3\mathbf{x}}_{\mathcal{E}_U} + \underbrace{\frac{1}{2\mu_0}\int \mathbf{B}^2 \mathrm{d}^3\mathbf{x}}_{\mathcal{E}_B} + \underbrace{\frac{1}{\gamma - 1}\int p \mathrm{d}^3\mathbf{x}}_{\mathcal{E}_T} \,,$$

$$\mathcal{E}_{\text{EPs}} = \underbrace{\iiint F \frac{1}{2} m_h v_\parallel^2 \mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x}}_{\mathcal{E}_{fv}} + \underbrace{\iiint F \mu B_\parallel \mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x}}_{\mathcal{E}_{fB}} \,.$$

- **Taking the sum of all time derivatives of energies yields**

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\text{total}} = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_U + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_B + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_T + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{fv} + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{fB} = 0 \,.$$

Energy-conserving coupling

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_U = \int \frac{1}{2}\frac{\partial \rho}{\partial t}\mathbf{U}^2 + \rho\frac{\partial \mathbf{U}}{\partial t}\cdot \mathbf{U}\mathrm{d}^3\mathbf{x}$$

Energy-conserving coupling

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_U = \int \frac{1}{2}\frac{\partial \rho}{\partial t}\mathbf{U}^2 + \rho\frac{\partial \mathbf{U}}{\partial t} \cdot \mathbf{U}\mathrm{d}^3\mathbf{x}$$
$$= \cdots - \int (\mathbf{J}_h + \nabla \times \mathbf{M}_h) \times \mathbf{B} \cdot \mathbf{U}\,\mathrm{d}^3\mathbf{x}$$

Energy-conserving coupling

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_U = \int \frac{1}{2}\frac{\partial \rho}{\partial t}\mathbf{U}^2 + \rho\frac{\partial \mathbf{U}}{\partial t}\cdot \mathbf{U}\mathrm{d}^3\mathbf{x}$$

$$= \cdots - \int (\mathbf{J}_h + \nabla \times \mathbf{M}_h) \times \mathbf{B}\cdot \mathbf{U}\,\mathrm{d}^3\mathbf{x}$$

$$= \cdots - \iiint m_h \frac{F}{B_\parallel^*} v_\parallel^2 (\nabla \times \mathbf{b}_0) \times \mathbf{B}\cdot \mathbf{U} - \frac{F}{B_\parallel^*}\mu(\mathbf{b}_0 \times \nabla B_\parallel) \times \mathbf{B}\cdot \mathbf{U} + F\mu(\nabla \times \mathbf{b}_0) \times \mathbf{B}\cdot \mathbf{U}\,\mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x}$$

Energy-conserving coupling

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_U = \int \frac{1}{2}\frac{\partial \rho}{\partial t}\mathbf{U}^2 + \rho\frac{\partial \mathbf{U}}{\partial t}\cdot\mathbf{U}\mathrm{d}^3\mathbf{x}$$
$$= \cdots - \int (\mathbf{J}_h + \nabla\times\mathbf{M}_h)\times\mathbf{B}\cdot\mathbf{U}\,\mathrm{d}^3\mathbf{x}$$
$$= \cdots - \iiint m_h\frac{F}{B_\parallel^*}v_\parallel^2(\nabla\times\mathbf{b}_0)\times\mathbf{B}\cdot\mathbf{U} - \frac{F}{B_\parallel^*}\mu(\mathbf{b}_0\times\nabla B_\parallel)\times\mathbf{B}\cdot\mathbf{U} + F\mu(\nabla\times\mathbf{b}_0)\times\mathbf{B}\cdot\mathbf{U}\,\mathrm{d}v_\parallel\mathrm{d}\mu\mathrm{d}^3\mathbf{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{fv} = \frac{m_h}{2}\iiint v_\parallel^2\frac{\partial F}{\partial t}\mathrm{d}v_\parallel\mathrm{d}\mu\mathrm{d}^3\mathbf{x}$$

Energy-conserving coupling

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_U &= \int \frac{1}{2}\frac{\partial \rho}{\partial t}\mathbf{U}^2 + \rho\frac{\partial \mathbf{U}}{\partial t}\cdot\mathbf{U}\mathrm{d}^3\mathbf{x} \\
&= \cdots - \int (\mathbf{J}_h + \nabla\times\mathbf{M}_h)\times\mathbf{B}\cdot\mathbf{U}\,\mathrm{d}^3\mathbf{x} \\
&= \cdots - \iiint m_h\frac{F}{B_\parallel^*}v_\parallel^2(\nabla\times\mathbf{b}_0)\times\mathbf{B}\cdot\mathbf{U} - \frac{F}{B_\parallel^*}\mu(\mathbf{b}_0\times\nabla B_\parallel)\times\mathbf{B}\cdot\mathbf{U} + F\mu(\nabla\times\mathbf{b}_0)\times\mathbf{B}\cdot\mathbf{U}\,\mathrm{d}v_\parallel\mathrm{d}\mu\mathrm{d}^3\mathbf{x}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{fv} &= \frac{m_h}{2}\iiint v_\parallel^2\frac{\partial F}{\partial t}\mathrm{d}v_\parallel\mathrm{d}\mu\mathrm{d}^3\mathbf{x} \\
&= -\frac{q_h}{2}\iiint v_\parallel^2\frac{\partial}{\partial v_\parallel}\left[\frac{F}{B_\parallel^*}\mathbf{B}^*\cdot\mathbf{E}^*\right]\mathrm{d}v_\parallel\mathrm{d}\mu\mathrm{d}^3\mathbf{x}
\end{aligned}
$$

Energy-conserving coupling

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_U = \int \frac{1}{2}\frac{\partial \rho}{\partial t}\mathbf{U}^2 + \rho\frac{\partial \mathbf{U}}{\partial t}\cdot\mathbf{U}\mathrm{d}^3\mathbf{x}$$
$$= \cdots - \int (\mathbf{J}_h + \nabla\times\mathbf{M}_h)\times\mathbf{B}\cdot\mathbf{U}\,\mathrm{d}^3\mathbf{x}$$
$$= \cdots - \iiint m_h\frac{F}{B^*_\parallel}v_\parallel^2(\nabla\times\mathbf{b}_0)\times\mathbf{B}\cdot\mathbf{U} - \frac{F}{B^*_\parallel}\mu(\mathbf{b}_0\times\nabla B_\parallel)\times\mathbf{B}\cdot\mathbf{U} + F\mu(\nabla\times\mathbf{b}_0)\times\mathbf{B}\cdot\mathbf{U}\,\mathrm{d}v_\parallel\mathrm{d}\mu\mathrm{d}^3\mathbf{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{fv} = \frac{m_h}{2}\iiint v_\parallel^2\frac{\partial F}{\partial t}\mathrm{d}v_\parallel\mathrm{d}\mu\mathrm{d}^3\mathbf{x}$$
$$= -\frac{q_h}{2}\iiint v_\parallel^2\frac{\partial}{\partial v_\parallel}\left[\frac{F}{B^*_\parallel}\mathbf{B}^*\cdot\mathbf{E}^*\right]\mathrm{d}v_\parallel\mathrm{d}\mu\mathrm{d}^3\mathbf{x}$$
$$= \cdots + m_h\iiint\frac{F}{B^*_\parallel}v_\parallel^2\mathbf{U}\cdot(\nabla\times\mathbf{b}_0)\times\mathbf{B}\,\mathrm{d}v_\parallel\mathrm{d}\mu\mathrm{d}^3\mathbf{x}$$

Energy-conserving coupling

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_U = \int \frac{1}{2}\frac{\partial \rho}{\partial t}\mathbf{U}^2 + \rho \frac{\partial \mathbf{U}}{\partial t}\cdot \mathbf{U}\mathrm{d}^3\mathbf{x}$$

$$= \cdots - \int (\mathbf{J}_h + \nabla \times \mathbf{M}_h) \times \mathbf{B}\cdot \mathbf{U}\,\mathrm{d}^3\mathbf{x}$$

$$= \cdots - \iiint \underbrace{m_h \frac{F}{B_\parallel^*} v_\parallel^2 (\nabla \times \mathbf{b}_0) \times \mathbf{B}\cdot \mathbf{U}}_{J_h\ \texttt{coupling}-\texttt{Curlb}} - \frac{F}{B_\parallel^*}\mu(\mathbf{b}_0 \times \nabla B_\parallel) \times \mathbf{B}\cdot\mathbf{U} + F\mu(\nabla \times \mathbf{b}_0)\times \mathbf{B}\cdot \mathbf{U}\,\mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{fv} = \frac{m_h}{2}\iiint v_\parallel^2 \frac{\partial F}{\partial t}\mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x}$$

$$= -\frac{q_h}{2}\iiint v_\parallel^2 \frac{\partial}{\partial v_\parallel}\left[\frac{F}{B_\parallel^*}\mathbf{B}^*\cdot \mathbf{E}^*\right]\mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x}$$

$$= \cdots + m_h \iiint \underbrace{\frac{F}{B_\parallel^*} v_\parallel^2 \mathbf{U}\cdot(\nabla \times \mathbf{b}_0)\times \mathbf{B}}_{J_h\ \texttt{coupling}-\texttt{Curlb}}\,\mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x}$$

Energy-conserving coupling

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_U = \int \frac{1}{2}\frac{\partial \rho}{\partial t}\mathbf{U}^2 + \rho\frac{\partial \mathbf{U}}{\partial t}\cdot \mathbf{U}\mathrm{d}^3\mathbf{x}$$

$$= \cdots - \int (\mathbf{J}_h + \nabla \times \mathbf{M}_h) \times \mathbf{B} \cdot \mathbf{U}\,\mathrm{d}^3\mathbf{x}$$

$$= \cdots - \iiint \underbrace{m_h \frac{F}{B_\parallel^*} v_\parallel^2 (\nabla \times \mathbf{b}_0) \times \mathbf{B} \cdot \mathbf{U}}_{J_h\ \texttt{coupling}-\texttt{Curlb}} - \frac{F}{B_\parallel^*}\mu(\mathbf{b}_0 \times \nabla B_\parallel) \times \mathbf{B} \cdot \mathbf{U} + F\mu(\nabla \times \mathbf{b}_0) \times \mathbf{B} \cdot \mathbf{U}\,\mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{fB} = \iiint \frac{\partial F}{\partial t}\mu B_\parallel + F\mu\frac{\partial B_\parallel}{\partial t}\mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x}$$

Energy-conserving coupling

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_U = \int \frac{1}{2}\frac{\partial \rho}{\partial t}\mathbf{U}^2 + \rho\frac{\partial \mathbf{U}}{\partial t}\cdot\mathbf{U}\mathrm{d}^3\mathbf{x}$$
$$= \cdots - \int (\mathbf{J}_h + \nabla\times\mathbf{M}_h)\times\mathbf{B}\cdot\mathbf{U}\,\mathrm{d}^3\mathbf{x}$$
$$= \cdots - \iiint \underbrace{m_h\frac{F}{B_\parallel^*}v_\parallel^2(\nabla\times\mathbf{b}_0)\times\mathbf{B}\cdot\mathbf{U}}_{J_h\ \texttt{coupling}-\texttt{Curlb}} - \frac{F}{B_\parallel^*}\mu(\mathbf{b}_0\times\nabla B_\parallel)\times\mathbf{B}\cdot\mathbf{U} + F\mu(\nabla\times\mathbf{b}_0)\times\mathbf{B}\cdot\mathbf{U}\,\mathrm{d}v_\parallel\mathrm{d}\mu\mathrm{d}^3\mathbf{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{fB} = \iiint \frac{\partial F}{\partial t}\mu B_\parallel + F\mu\frac{\partial B_\parallel}{\partial t}\mathrm{d}v_\parallel\mathrm{d}\mu\mathrm{d}^3\mathbf{x}$$
$$= -\iiint \frac{\partial}{\partial \mathbf{x}}\left[\frac{F}{B_\parallel^*}(v_\parallel\mathbf{B}^* - \mathbf{b}_0\times\mathbf{E}^*)\right]\mu B_\parallel + F\mu\mathbf{b}_0\cdot\nabla\times(\mathbf{B}\times\mathbf{U})\mathrm{d}v_\parallel\mathrm{d}\mu\mathrm{d}^3\mathbf{x}$$

Energy-conserving coupling

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_U = \int \frac{1}{2}\frac{\partial \rho}{\partial t}\mathbf{U}^2 + \rho \frac{\partial \mathbf{U}}{\partial t} \cdot \mathbf{U} \mathrm{d}^3\mathbf{x}$$

$$= \cdots - \int (\mathbf{J}_h + \nabla \times \mathbf{M}_h) \times \mathbf{B} \cdot \mathbf{U}\, \mathrm{d}^3\mathbf{x}$$

$$= \cdots - \iiint \underbrace{m_h \frac{F}{B^*_\parallel} v_\parallel^2 (\nabla \times \mathbf{b}_0) \times \mathbf{B} \cdot \mathbf{U}}_{J_h\ \mathtt{coupling-Curlb}} - \frac{F}{B^*_\parallel}\mu(\mathbf{b}_0 \times \nabla B_\parallel) \times \mathbf{B} \cdot \mathbf{U} + F\mu(\nabla \times \mathbf{b}_0) \times \mathbf{B} \cdot \mathbf{U}\, \mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{fB} = \iiint \frac{\partial F}{\partial t}\mu B_\parallel + F\mu \frac{\partial B_\parallel}{\partial t} \mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x}$$

$$= -\iiint \frac{\partial}{\partial \mathbf{x}} \left[\frac{F}{B^*_\parallel}(v_\parallel \mathbf{B}^* - \mathbf{b}_0 \times \mathbf{E}^*) \right] \mu B_\parallel + F\mu \mathbf{b}_0 \cdot \nabla \times (\mathbf{B} \times \mathbf{U}) \mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x}$$

$$= \cdots + \iiint \frac{F}{B^*_\parallel}\mathbf{U} \cdot \mathbf{B} \times (\nabla B_\parallel \times \mathbf{b}_0) - F\mu \mathbf{b}_0 \cdot \nabla \times (\mathbf{B} \times \mathbf{U}) \mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x}$$

Energy-conserving coupling

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_U = \int \frac{1}{2}\frac{\partial \rho}{\partial t}\mathbf{U}^2 + \rho\frac{\partial \mathbf{U}}{\partial t}\cdot \mathbf{U}\mathrm{d}^3\mathbf{x}$$

$$= \cdots - \int (\mathbf{J}_h + \nabla \times \mathbf{M}_h) \times \mathbf{B}\cdot\mathbf{U}\,\mathrm{d}^3\mathbf{x}$$

$$= \cdots - \iiint \underbrace{m_h \frac{F}{B_\parallel^*} v_\parallel^2 (\nabla \times \mathbf{b}_0) \times \mathbf{B}\cdot\mathbf{U}}_{J_h\ \texttt{coupling}-\texttt{Curlb}} - \underbrace{\frac{F}{B_\parallel^*}\mu(\mathbf{b}_0 \times \nabla B_\parallel)\times \mathbf{B}\cdot\mathbf{U}}_{J_h\ \texttt{coupling}-\texttt{GradB}} + \underbrace{F\mu(\nabla\times\mathbf{b}_0)\times\mathbf{B}\cdot\mathbf{U}\,\mathrm{d}v_\parallel}_{M_h\ \texttt{coupling}}\mathrm{d}\mu\mathrm{d}^3\mathbf{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{fB} = \iiint \frac{\partial F}{\partial t}\mu B_\parallel + F\mu\frac{\partial B_\parallel}{\partial t}\mathrm{d}v_\parallel\mathrm{d}\mu\mathrm{d}^3\mathbf{x}$$

$$= -\iiint \frac{\partial}{\partial \mathbf{x}}\left[\frac{F}{B_\parallel^*}(v_\parallel \mathbf{B}^* - \mathbf{b}_0 \times \mathbf{E}^*)\right]\mu B_\parallel + F\mu\mathbf{b}_0\cdot\nabla\times(\mathbf{B}\times\mathbf{U})\mathrm{d}v_\parallel\mathrm{d}\mu\mathrm{d}^3\mathbf{x}$$

$$= \cdots + \iiint \underbrace{\frac{F}{B_\parallel^*}\mathbf{U}\cdot\mathbf{B}\times(\nabla B_\parallel \times \mathbf{b}_0)}_{J_h\ \texttt{coupling}-\texttt{GradB}} - \underbrace{F\mu\mathbf{b}_0\cdot\nabla\times(\mathbf{B}\times\mathbf{U})}_{M_h\ \texttt{coupling}}\mathrm{d}v_\parallel\mathrm{d}\mu\mathrm{d}^3\mathbf{x}$$

Linear perturbation

- **Linearized MHD part with a zero-flow equilibrium:**

$$\rho = \rho_0 + \tilde{\rho}, \quad \mathbf{B} = \mathbf{B}_0 + \tilde{\mathbf{B}}, \quad \mathbf{U} = \tilde{\mathbf{U}}, \quad p = p_0 + \tilde{p}\,.$$

Linear MHD:
$$\begin{cases} \rho_0 \dfrac{\partial \tilde{\mathbf{U}}}{\partial t} = \dfrac{1}{\mu_0}(\nabla \times \mathbf{B}_0) \times \tilde{\mathbf{B}} + \dfrac{1}{\mu_0}(\nabla \times \tilde{\mathbf{B}}) \times \mathbf{B}_0 + (q_h n_h \mathbf{U} - \mathbf{J}_h - \nabla \times \mathbf{M}_h) \times \mathbf{B} - \nabla \tilde{p}\,, \\ \dfrac{\partial \tilde{\mathbf{B}}}{\partial t} + \nabla \times (\mathbf{B}_0 \times \tilde{\mathbf{U}}) = 0\,, \\ \dfrac{\partial \tilde{p}}{\partial t} + \nabla \cdot (p_0 \tilde{\mathbf{U}}) + (\gamma - 1) p_0 \nabla \cdot \tilde{\mathbf{U}} = 0\,. \end{cases}$$

Linear perturbation

- **Linearized MHD part with a zero-flow equilibrium:**

$$\rho = \rho_0 + \tilde{\rho}, \quad \mathbf{B} = \mathbf{B}_0 + \tilde{\mathbf{B}}, \quad \mathbf{U} = \tilde{\mathbf{U}}, \quad p = p_0 + \tilde{p}\,.$$

$$\text{Linear MHD:} \begin{cases} \rho_0 \dfrac{\partial \tilde{\mathbf{U}}}{\partial t} = \dfrac{1}{\mu_0}(\nabla \times \mathbf{B}_0) \times \tilde{\mathbf{B}} + \dfrac{1}{\mu_0}(\nabla \times \tilde{\mathbf{B}}) \times \mathbf{B}_0 + (q_h n_h \mathbf{U} - \mathbf{J}_h - \nabla \times \mathbf{M}_h) \times \mathbf{B} - \nabla \tilde{p}\,, \\ \dfrac{\partial \tilde{\mathbf{B}}}{\partial t} + \nabla \times (\mathbf{B}_0 \times \tilde{\mathbf{U}}) = 0\,, \\ \dfrac{\partial \tilde{p}}{\partial t} + \nabla \cdot (p_0 \tilde{\mathbf{U}}) + (\gamma - 1) p_0 \nabla \cdot \tilde{\mathbf{U}} = 0\,. \end{cases}$$

- **Due to the linearization, $\mathcal{E}_{\text{total}}$ is no longer conserved.**

$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\text{total}} = \int (\nabla \times B_0 + \nabla \times \mathbf{M}_h) \times \tilde{\mathbf{B}} \cdot \tilde{\mathbf{U}} - (p_0 - \tilde{p}) \nabla \cdot \tilde{\mathbf{U}}\, \mathrm{d}^3\mathbf{x}$.

$\Rightarrow$ However, still can describe **Alfvén eigenmodes** and their **linear** and **non-linear** interactions with EPs.

$\Rightarrow$ Exact energy conservation can be recovered by calculating **B** and p as an initial-value problem.

Time splitting

- **Split the whole system into 4 Hamiltonian (skew-symmetric) sub-steps + 1 non-Hamiltonian sub-step.**

1. `Driftkinetic` $\Phi^1 : (\mathbf{x}^n, v_\parallel^n) \rightarrow (\mathbf{x}^{n+1}, v_\parallel^{n+1})$
2. `Shear Alfvén` + M_h `coupling` $\Phi^2 : (\mathbf{u}^n, \mathbf{b}^n) \rightarrow (\mathbf{u}^{n+1}, \mathbf{b}^{n+1})$
3. J_h `coupling` - `Curlb` $\Phi^3 : (\mathbf{u}^n, v_\parallel^n) \rightarrow (\mathbf{u}^{n+1}, v_\parallel^{n+1})$
4. J_h `coupling` - `GradB` $\Phi^4 : (\mathbf{u}^n, \mathbf{x}^n) \rightarrow (\mathbf{u}^{n+1}, \mathbf{x}^{n+1})$
5. `Magnetosonic` + `non-Hamiltonian` $\Phi^5 : (\mathbf{u}^n, p^n) \rightarrow (\mathbf{u}^{n+1}, p^{n+1})$

- **Each sub-steps are combined via *Lie-Trotter* or *Strang* splitting methods.**

Lie-Trotter : $\Phi^L_{\Delta t} := \Phi^5_{\Delta t} \circ \Phi^4_{\Delta t} \circ \Phi^3_{\Delta t} \circ \Phi^2_{\Delta t} \circ \Phi^1_{\Delta t}$

Strang : $\Phi^S_{\Delta t} := \Phi^L_{\Delta t/2} \circ (\Phi^L_{\Delta t/2})^{-1}$

STEP 1: Driftkinetic

$$
\begin{cases}
\dfrac{\partial F}{\partial t} + \dfrac{\partial}{\partial \mathbf{x}} \cdot \left[\dfrac{F}{B^*_\parallel} (v_\parallel \mathbf{B}^* - \mathbf{b}_0 \times \mathbf{E}^*) \right] + \dfrac{q_h}{m_h} \dfrac{\partial}{\partial v_\parallel} \left(\dfrac{F}{B^*_\parallel} \mathbf{B}^* \cdot \mathbf{E}^* \right) = 0 \,, \\
\mathbf{B}^* = \mathbf{B} + \dfrac{m_h}{q_h} v_\parallel \nabla \times \mathbf{b}_0 \,, \quad \mathbf{E}^* = -\mathbf{U} \times \mathbf{B} - \dfrac{\mu}{q_h} \nabla B_\parallel \,.
\end{cases}
$$

Driftkinetic

$$
\begin{cases}
\dot{\mathbf{x}} = v_\parallel \dfrac{\mathbf{B}^*}{B^*_\parallel} + \epsilon \dfrac{1}{B^*_\parallel} \mathbf{b}_0 \times \mu \nabla B_\parallel \,, \\
\dot{v}_\parallel = - \dfrac{\mathbf{B}^*}{B^*_\parallel} \cdot \mu \nabla B_\parallel \,,
\end{cases}
$$

$$
\mathcal{E}_{\text{EPs}}(\mathbf{x}, v_\parallel) = \iiint F \frac{1}{2} m_h v_\parallel^2 \mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x} + \iiint F \mu B_\parallel(\mathbf{x}) \mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x} \,.
$$

Discrete gradient method

- **Driftkinetic equations have the following ODE form**

$$\dot{\mathbf{z}} = \mathbb{S}(\mathbf{z})\,\nabla\mathcal{I}(\mathbf{z})\,,$$

i.e. $\mathbf{z} = \{\mathbf{x}, v_\parallel\}\,,\quad \mathcal{I}(\mathbf{z}) = \mu B_\parallel(\mathbf{x}) + \frac{1}{2}v_\parallel^2,$

```
Driftkinetic
```

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{v}_\parallel\end{bmatrix} = \begin{bmatrix}\epsilon/B_\parallel^*\,\mathbf{b}_0\times & 1/B_\parallel^*\,\mathbf{B}^*(\mathbf{x},v_\parallel)\,\cdot\\ -1/B_\parallel^*\,\mathbf{B}^*(\mathbf{x},v_\parallel)\,\cdot & 0\end{bmatrix}\begin{bmatrix}\mu\nabla B_\parallel\\ v_\parallel\end{bmatrix}.$$

- **Then the ODE is discretized as**

$$\frac{\mathbf{z}_{n+1}-\mathbf{z}_n}{h} = \tilde{\mathbb{S}}(\mathbf{z}_{n+1},\mathbf{z}_n)\bar{\nabla}\mathcal{I}(\mathbf{z}_{n+1},\mathbf{z}_n)\,,$$

$\tilde{\mathbb{S}}$ is any **skew-symmetric matrix** s.t. $\quad \lim_{\mathbf{z}_{n+1}\to\mathbf{z}_n}\tilde{\mathbb{S}}(\mathbf{z}_{n+1},\mathbf{z}_n) = \mathbb{S}(\mathbf{z}_n)\,.$

$\bar{\nabla}\mathcal{I}(\mathbf{z}_{n+1},\mathbf{z}_n)$ is a **discrete gradient** s.t.

$$\begin{cases}\bar{\nabla}\mathcal{I}(\mathbf{z}_{n+1},\mathbf{z}_n) = \dfrac{\mathcal{I}(\mathbf{z}_{n+1})-\mathcal{I}(\mathbf{z}_n)}{\mathbf{z}_{n+1}-\mathbf{z}_n}\,,\\ \bar{\nabla}\mathcal{I}(\mathbf{z}_n,\mathbf{z}_n) = \nabla\mathcal{I}(\mathbf{z}_n)\,.\end{cases}$$

Discrete gradient method

- **Mid-point $\tilde{\mathbb{S}}$**

$$\tilde{\mathbb{S}}(\mathbf{z}_{n+1}, \mathbf{z}_n) := \mathbb{S}(\frac{\mathbf{z}_{n+1} + \mathbf{z}_n}{2}) \,.$$

- **Gozalez (mid-point) discrete gradient $\bar{\nabla}\mathcal{I}$** [2]

$$\bar{\nabla}\mathcal{I}(\mathbf{z}_{n+1}, \mathbf{z}_n) = \nabla\mathcal{I}(\mathbf{z}_{n+1/2}) + (\mathbf{z}_{n+1} - \mathbf{z}_n)\frac{\mathcal{I}(\mathbf{z}_{n+1}) - \mathcal{I}(\mathbf{z}_n) - (\mathbf{z}_{n+1} - \mathbf{z}_n)\cdot\nabla\mathcal{I}(\mathbf{z}_{n+1/2})}{||\mathbf{z}_{n+1} - \mathbf{z}_n||^2} \,,$$

where $\mathbf{z}_{n+1/2} = (\mathbf{z}_{n+1} + \mathbf{z}_n)/2$.

- **Fixed-point iteration**

$$\begin{aligned}
\mathbf{z}^0_{n+1} &= \mathbf{z}_n + h\,\mathbb{S}(z_n)\nabla\mathcal{I}(\mathbf{z}_n)\,,\\
\mathbf{z}^1_{n+1} &= \mathbf{z}_n + h\,\tilde{\mathbb{S}}(\mathbf{z}^0_{n+1}, \mathbf{z}_n)\bar{\nabla}\mathcal{I}(\mathbf{z}^0_{n+1}, \mathbf{z}_n)\,,\\
&\dots\\
\mathbf{z}^k_{n+1} &= \mathbf{z}_n + h\,\tilde{\mathbb{S}}(\mathbf{z}^{k-1}_{n+1}, \mathbf{z}_n)\bar{\nabla}\mathcal{I}(\mathbf{z}^{k-1}_{n+1}, \mathbf{z}_n)\,,
\end{aligned}$$

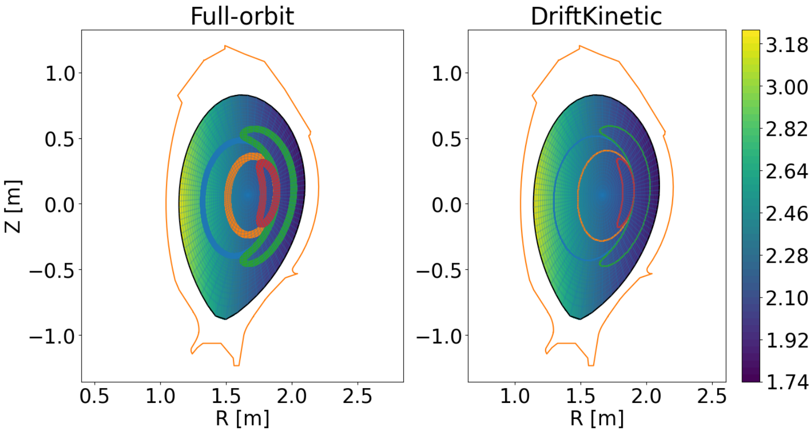
for $k = 0, 1, \dots$ until satisfying

$$||\mathbf{z}^k_{n+1} - \mathbf{z}^{k-1}_{n+1}|| < \text{tolerance}\,.$$

[2] O. Gonzalez (1996) "Time integration and discrete hamiltonian systems"

Discrete gradient method

- **Describe passing, co-passing, trapped, co-trapped particles.**



Discrete gradient method

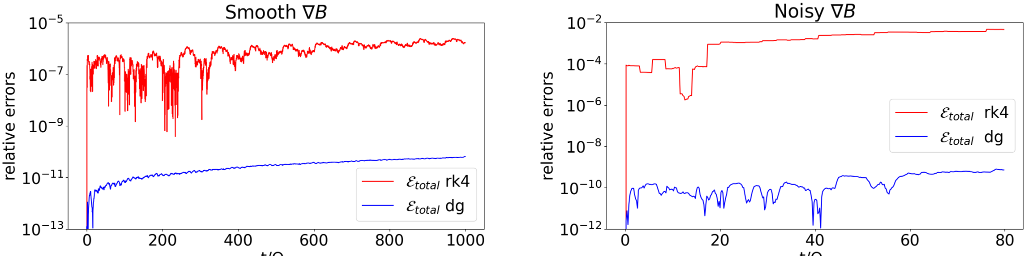
- **Discrete gradient method preserves $\mathcal{I}(\mathbf{z})$.**

  By the definition of discrete gradient,

$$\mathcal{I}(\mathbf{z}_{n+1}) - \mathcal{I}(\mathbf{z}_n) = \bar{\nabla}\mathcal{I}(\mathbf{z}_{n+1}, \mathbf{z}_n)^\top (\mathbf{z}_{n+1} - \mathbf{z}_n)\,.$$

  By using the discrete gradient update scheme $\mathbf{z}_{n+1} - \mathbf{z}_n = h\tilde{\mathbb{S}}(\mathbf{z}_{n+1} - \mathbf{z}_n)\bar{\nabla}\mathcal{I}(\mathbf{z}_{n+1} - \mathbf{z}_n)$,

$$\mathcal{I}(\mathbf{z}_{n+1}) - \mathcal{I}(\mathbf{z}_n) = h\bar{\nabla}\mathcal{I}(\mathbf{z}_{n+1}, \mathbf{z}_n)^\top \underbrace{\tilde{\mathbb{S}}(\mathbf{z}_{n+1} - \mathbf{z}_n)}_{\text{skew-symmetric}} \bar{\nabla}\mathcal{I}(\mathbf{z}_{n+1} - \mathbf{z}_n) = 0\,.$$



STEP 2: Shear Alfvén + M_h coupling

`ShearAlfvén` + M_h `coupling`

$$\begin{cases} \rho_0 \dfrac{\partial \tilde{\mathbf{U}}}{\partial t} = (\nabla \times \tilde{\mathbf{B}} - \nabla \times \mathbf{M}_h) \times \mathbf{B}_0 \,, \\ \dfrac{\partial \tilde{\mathbf{B}}}{\partial t} = -\nabla \times (\mathbf{B}_0 \times \tilde{\mathbf{U}}) \,, \end{cases} \qquad \mathcal{E}_{\mathrm{SA}}(\tilde{\mathbf{U}}, \tilde{\mathbf{B}}) = \frac{1}{2} \int \rho \tilde{\mathbf{U}}^2 \mathrm{d}^3\mathbf{x} + \frac{1}{2} \int \mathbf{B}^2 \mathrm{d}^3\mathbf{x} + \iiint F \mu B_\parallel \mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x} \,.$$

- FE coefficients update for $\tilde{\mathbf{U}} = \mathbf{u}^\top \vec{\mathbf{\Lambda}}^2 \in V_h^2$ and $\tilde{\mathbf{B}} = \mathbf{b}^\top \vec{\mathbf{\Lambda}}^2 \in V_h^2$:

$$\begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{b}} \end{bmatrix} = \frac{\Delta t}{2} \begin{bmatrix} 0 & (\mathbb{M}^2_\rho)^{-1} \mathcal{T}^\top \mathbb{C}^\top \\ -\mathbb{C}\mathcal{T}(\mathbb{M}^2_\rho)^{-1} & 0 \end{bmatrix} \begin{bmatrix} \mathbb{M}^2_\rho \mathbf{u} \\ \mathbb{M}^2 \mathbf{b} + \mathcal{P}_b^\top \sum_k^{N_p} \omega_k \mu_k \mathbf{\Lambda}^0(\boldsymbol{\eta}_k) \end{bmatrix} ,$$

where

$$\mathbb{M}^2 := \int (\vec{\mathbf{\Lambda}}^2)^\top G \vec{\mathbf{\Lambda}}^2 \frac{1}{\sqrt{g}} \mathrm{d}\boldsymbol{\eta} \,, \qquad \mathbb{M}^2_\rho := \int \rho_0(\boldsymbol{\eta}) (\vec{\mathbf{\Lambda}}^2)^\top G \vec{\mathbf{\Lambda}}^2 \frac{1}{\sqrt{g}} \mathrm{d}\boldsymbol{\eta} \,,$$

$$\mathcal{T} := \hat{\mathbf{\Pi}}^1 \left[\frac{1}{\sqrt{g}} \mathbf{B}_0^2 \times \vec{\mathbf{\Lambda}}^2 \right] , \qquad \mathcal{P}_b := \hat{\mathbf{\Pi}}^0 \left[\frac{1}{\sqrt{g}} \mathbf{b}_0^1 \cdot \vec{\mathbf{\Lambda}}^2 \right] , \qquad B_\parallel = \mathcal{P}_b \mathbf{b}\, \mathbf{\Lambda}^0 \,.$$

STEP 2: Shear Alfvén + M_h coupling

ShearAlfvén + M_h coupling

$$\begin{cases} \rho_0 \dfrac{\partial \tilde{\mathbf{U}}}{\partial t} = (\nabla \times \tilde{\mathbf{B}} - \nabla \times \mathbf{M}_h) \times \mathbf{B}_0 \,, \\ \dfrac{\partial \tilde{\mathbf{B}}}{\partial t} = -\nabla \times (\mathbf{B}_0 \times \tilde{\mathbf{U}}) \,, \end{cases} \qquad \mathcal{E}_{\mathrm{SA}}(\tilde{\mathbf{U}}, \tilde{\mathbf{B}}) = \frac{1}{2} \int \rho \tilde{\mathbf{U}}^2 \mathrm{d}^3\mathbf{x} + \frac{1}{2} \int \mathbf{B}^2 \mathrm{d}^3\mathbf{x} + \iiint F \mu B_\parallel \mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x} \,.$$

- FE coefficients update for $\tilde{\mathbf{U}} = \mathbf{u}^\top \vec{\mathbf{\Lambda}}^2 \in V_h^2$ and $\tilde{\mathbf{B}} = \mathbf{b}^\top \vec{\mathbf{\Lambda}}^2 \in V_h^2$:

$$\begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{b}} \end{bmatrix} = \frac{\Delta t}{2} \begin{bmatrix} 0 & (\mathbb{M}^2_\rho)^{-1} \mathcal{T}^\top \mathbb{C}^\top \\ -\mathbb{C} \mathcal{T} (\mathbb{M}^2_\rho)^{-1} & 0 \end{bmatrix} \begin{bmatrix} \mathbb{M}^2_\rho \mathbf{u} \\ \mathbb{M}^2 \mathbf{b} + \mathcal{P}_b^\top \sum_k^{N_p} \omega_k \mu_k \mathbf{\Lambda}^0(\boldsymbol{\eta}_k) \end{bmatrix} ,$$

With semi-implicit Crank-Nicolson method, following discrete energy is conserved

$$\mathcal{E}_{h,\mathrm{SA}} = \frac{1}{2} \mathbf{u}^\top \mathbb{M}^2_\rho \mathbf{u} + \frac{1}{2} \mathbf{b}^\top \mathbb{M}^2 \mathbf{b} + \mathbf{b}^\top \mathcal{P}_b^\top \sum_k^{N_p} \omega_k \mu_k \mathbf{\Lambda}^0(\boldsymbol{\eta}_p) \,.$$

STEP 3: J_h coupling - Curlb

J_h coupling - Curlb

$$\begin{cases} \rho_0 \dfrac{\partial \tilde{\mathbf{U}}}{\partial t} = -\displaystyle\iint F \frac{1}{B_\parallel^*} v_\parallel^2 \nabla \times \mathbf{b}_0 \mathrm{d}v_\parallel \mathrm{d}\mu \times \mathbf{B}\,, \\ \dfrac{\partial v_\parallel}{\partial t} = -\dfrac{1}{B_\parallel^*} v_\parallel (\nabla \times \mathbf{b}_0) \cdot (\mathbf{B} \times \tilde{\mathbf{U}})\,, \end{cases} \qquad \mathcal{E}_{\text{CC Curlb}} = \frac{1}{2}\int \rho \tilde{\mathbf{U}}^2 \mathrm{d}^3\mathbf{x} + \iiint F \frac{1}{2} v_\parallel^2 \mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x}\,.$$

- FE coefficient update for $\tilde{\mathbf{U}} = \mathbf{u}^\top \vec{\Lambda}^2 \in V_h^2$ and particle pushing $v_\parallel$:

$$\begin{bmatrix} \dot{\mathbf{u}} \\ \dot{V}_\parallel \end{bmatrix} = \frac{\Delta t}{2} \begin{bmatrix} 0 & (\mathbb{M}_\rho^2)^{-1} \mathbb{\Lambda}^{2\top} \mathbb{B}_\parallel^{*,-1} \mathbb{V}_\parallel \sqrt{\mathbb{g}}^{-1} \sqrt{\mathbb{g}}^{-1} \mathbb{B}^\times \mathbb{b}_0^{\nabla\times} \\ -\mathbb{b}_0^{\nabla\times\top} \mathbb{B}^\times \sqrt{\mathbb{g}}^{-1} \sqrt{\mathbb{g}}^{-1} \mathbb{V}_\parallel \mathbb{B}_\parallel^{*,-1} \mathbb{\Lambda}^2 (\mathbb{M}_\rho^2)^{-1} & 0 \end{bmatrix} \begin{bmatrix} \mathbb{M}_\rho^2 \mathbf{u} \\ \mathbb{W} \bar{V}_\parallel \end{bmatrix},$$

where $V_\parallel := (v_{\parallel,1}, \cdots v_{\parallel,N_p})^\top \in \mathbb{R}^{N_p}$, $\bar{V}_\parallel := (V_\parallel, V_\parallel, V_\parallel) \in \mathbb{R}^{3N_p}$ and $\mathbb{B}^\times$, $\mathbb{b}_0^{\nabla\times}$, $\mathbb{\Lambda}^2$ are the block matrices which are diagonally staked collocation vectors. e.g.

$$\mathbb{\Lambda}^2 := \mathrm{diag}(\mathbb{\Lambda}_1^2, \mathbb{\Lambda}_2^2, \mathbb{\Lambda}_3^2) \in \mathbb{R}^{3N_p \times 3N_p}$$

$$\mathbb{\Lambda}_\mu^2 := (\Lambda_{\mu,i}^n(\boldsymbol{\eta}_k))_{0 \le i \le N_\mu^n,\, 1 \le k \le N_p,\, n \in \{v,1,2\},\, \mu \in \{1,2,3\}} \in \mathbb{R}^{3N_p}$$

STEP 3: J_h coupling - Curlb

J_h coupling - Curlb

$$\begin{cases} \rho_0 \dfrac{\partial \tilde{\mathbf{U}}}{\partial t} = -\displaystyle\iint F \frac{1}{B_\parallel^*} v_\parallel^2 \nabla \times \mathbf{b}_0 \mathrm{d}v_\parallel \mathrm{d}\mu \times \mathbf{B}\,, \\ \dfrac{\partial v_\parallel}{\partial t} = -\dfrac{1}{B_\parallel^*} v_\parallel (\nabla \times \mathbf{b}_0) \cdot (\mathbf{B} \times \tilde{\mathbf{U}})\,, \end{cases} \qquad \mathcal{E}_{\text{CC Curlb}} = \frac{1}{2}\int \rho \tilde{\mathbf{U}}^2 \mathrm{d}^3\mathbf{x} + \iiint F \frac{1}{2} v_\parallel^2 \mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x}\,.$$

- FE coefficient update for $\tilde{\mathbf{U}} = \mathbf{u}^\top \vec{\boldsymbol{\Lambda}}^2 \in V_h^2$ and particle pushing $v_\parallel$:

$$\begin{bmatrix} \dot{\mathbf{u}} \\ \dot{V}_\parallel \end{bmatrix} = \frac{\Delta t}{2} \begin{bmatrix} 0 & (\mathbb{M}_\rho^2)^{-1} \mathbb{\Lambda}^{2\,\top} \mathbb{B}_\parallel^{*,-1} \mathbb{V}_\parallel \sqrt{\mathbb{g}}^{-1} \sqrt{\mathbb{g}}^{-1} \mathbb{B}^\times \mathbb{b}_0^{\nabla\times} \\ -\mathbb{b}_0^{\nabla\times\,\top} \mathbb{B}^\times \sqrt{\mathbb{g}}^{-1} \sqrt{\mathbb{g}}^{-1} \mathbb{V}_\parallel \mathbb{B}_\parallel^{*,-1} \mathbb{\Lambda}^2 (\mathbb{M}_\rho^2)^{-1} & 0 \end{bmatrix} \begin{bmatrix} \mathbb{M}_\rho^2 \mathbf{u} \\ \mathbb{W} \bar{V}_\parallel \end{bmatrix},$$

With semi-implicit Crank-Nicolson method, following discrete energy is conserved

$$\mathcal{E}_{h,\text{CC Curlb}} = \frac{1}{2}\mathbf{u}^\top \mathbb{M}_\rho^2 \mathbf{u} + \sum_k^{N_p} \omega_k v_{\parallel,k}^2\,.$$

STEP 4: J_h coupling - GradB

J_h coupling - GradB

$$\begin{cases} \rho_0 \dfrac{\partial \tilde{\mathbf{U}}}{\partial t} = \displaystyle\iint F \frac{1}{B_\parallel^*} \mathbf{b}_0 \times (\mu \nabla B_\parallel) \mathrm{d}v_\parallel \mathrm{d}\mu \times \mathbf{B}\,, \\ \dfrac{\partial \mathbf{x}}{\partial t} = \dfrac{1}{B_\parallel^*} \mathbf{b}_0 \times \tilde{\mathbf{U}} \times \mathbf{B}\,, \end{cases} \qquad \mathcal{E}_{\text{CC GradB}} = \frac{1}{2} \int \rho \tilde{\mathbf{U}}^2 \mathrm{d}^3\mathbf{x} + \iiint F \mu B_\parallel(\mathbf{x}) \mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x}\,.$$

- FE coefficient update for $\tilde{\mathbf{U}} = \mathbf{u}^\top \vec{\boldsymbol{\Lambda}}^2 \in V_h^2$ and particle pushing $\mathbf{x}$:

$$\begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{H}} \end{bmatrix} = \frac{\Delta t}{2} \cdot \begin{bmatrix} 0 & (\mathbb{M}_\rho^2)^{-1} \mathbb{\Lambda}^{2\top} \mathbb{B}_\parallel^{*,-1} \sqrt{\mathbb{g}}^{-1} \mathbb{B}^\times \bar{G}^{-1} \mathbb{b}_0^\times \bar{G}^{-1} \\ -\bar{G}^{-1} \mathbb{b}_0^\times \bar{G}^{-1} \mathbb{B}^\times \sqrt{\mathbb{g}}^{-1} \mathbb{B}_\parallel^{*,-1} \mathbb{\Lambda}^2 (\mathbb{M}_\rho^2)^{-1} & 0 \end{bmatrix} \begin{bmatrix} \mathbb{M}_\rho^2 \mathbf{u} \\ \mathbb{W} \bar{\mu} \mathbb{\Lambda}^1 \mathbb{G} \mathcal{P}_b \mathbf{b} \end{bmatrix},$$

where $\mathbf{H} := (\eta_{1,1}, \ldots, \eta_{N_p,1}, \eta_{1,2}, \ldots, \eta_{N_p,2}, \eta_{1,3}, \ldots, \eta_{N_p,3})^\top \in \mathbb{R}^{3N_p}$.

STEP 4: J_h coupling - GradB

J_h coupling - GradB

$$\begin{cases} \rho_0 \dfrac{\partial \tilde{\mathbf{U}}}{\partial t} = \displaystyle\iint F \frac{1}{B^*_\parallel} \mathbf{b}_0 \times (\mu \nabla B_\parallel) \mathrm{d}v_\parallel \mathrm{d}\mu \times \mathbf{B}\,, \\ \dfrac{\partial \mathbf{x}}{\partial t} = \dfrac{1}{B^*_\parallel} \mathbf{b}_0 \times \tilde{\mathbf{U}} \times \mathbf{B}\,, \end{cases}$$

$$\mathcal{E}_{\text{CC GradB}} = \frac{1}{2} \int \rho \tilde{\mathbf{U}}^2 \mathrm{d}^3\mathbf{x} + \iiint F \mu B_\parallel(\mathbf{x}) \mathrm{d}v_\parallel \mathrm{d}\mu \mathrm{d}^3\mathbf{x}\,.$$

- FE coefficient update for $\tilde{\mathbf{U}} = \mathbf{u}^\top \vec{\mathbf{\Lambda}}^2 \in V_h^2$ and particle pushing $\mathbf{x}$:

$$\begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{H}} \end{bmatrix} = \frac{\Delta t}{2} \cdot \begin{bmatrix} 0 & (\mathbb{M}^2_\rho)^{-1} {\mathbb{\Lambda}^2}^\top \mathbb{B}^{*,-1}_\parallel \sqrt{\mathbb{g}}^{-1} \mathbb{B}^\times \bar{G}^{-1} \mathbb{b}_0^\times \bar{G}^{-1} \\ -\bar{G}^{-1} \mathbb{b}_0^\times \bar{G}^{-1} \mathbb{B}^\times \sqrt{\mathbb{g}}^{-1} \mathbb{B}^{*,-1}_\parallel \mathbb{\Lambda}^2 (\mathbb{M}^2_\rho)^{-1} & 0 \end{bmatrix} \begin{bmatrix} \mathbb{M}^2_\rho \mathbf{u} \\ \mathbb{W} \bar{\mu} \mathbb{\Lambda}^1 \mathbb{G} \mathcal{P}_b \mathbf{b} \end{bmatrix},$$

The following discrete energy of the system is not exactly conserved due to the **non-quadratic** energy term:

$$\mathcal{E}_{h,\text{CC GradB}} = \frac{1}{2} \mathbf{u}^\top \mathbb{M}^2_\rho \mathbf{u} + \mathbf{b}^\top \mathcal{P}_b^\top \sum_k^{N_p} \omega_k \mu_k \mathbf{\Lambda}^0(\boldsymbol{\eta}_k)\,.$$

STEP 5: Magnetosonic + non-Hamiltonian

Magnetosonic + non-Hamiltonian

$$\begin{cases} \rho_0 \dfrac{\partial \tilde{\mathbf{U}}}{\partial t} = -\nabla \tilde{p} + (\nabla \times \mathbf{B}_0 - \nabla \times \mathbf{M}_h) \times \tilde{\mathbf{B}} \,, \\ \dfrac{\partial \tilde{p}}{\partial t} = -\nabla \cdot (p_0 \tilde{\mathbf{U}}) \,, \end{cases} \qquad \mathcal{E}_{\text{MS}}(\tilde{\mathbf{U}}, \tilde{p}) = \frac{1}{2} \int \rho \tilde{\mathbf{U}}^2 \mathrm{d}^3\mathbf{x} + \frac{1}{\gamma - 1} \int \tilde{p} \mathrm{d}^3\mathbf{x} \,.$$

- FE coefficients update for $\tilde{\mathbf{U}} = \mathbf{u}^\top \vec{\mathbf{\Lambda}}^2 \in V_h^2$ and $\tilde{p} = \mathbf{p}^\top \mathbf{\Lambda}^3 \in V_h^3$:

$$\begin{bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{p}} \end{bmatrix} = \frac{\Delta t}{2} \begin{bmatrix} 0 & (\mathbb{M}^2_\rho)^{-1} {\mathcal{U}^2}^\top \mathbb{D}^\top \mathbb{M}_3 \\ -\mathbb{D}\mathcal{S}^2 - (\gamma - 1)\mathcal{K}^2 \mathbb{D} \mathcal{U}^2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix}$$

$$+ \Delta t (\mathbb{M}^2_\rho)^{-1} \begin{bmatrix} \mathbb{M}^2_J \mathbf{b} + \sum_k^{N_p} \omega_k \mu_k \left\{ (\hat{\nabla} \times \hat{\mathbf{b}}^1_0) \times \hat{\mathbf{B}}^2 \right\} (\boldsymbol{\eta}_k) \\ 0 \end{bmatrix} .$$

ITPA linear benchmark

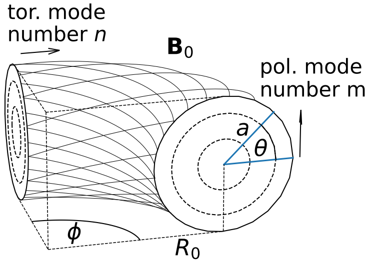
- **ITPA linear benchmark case**[a]:

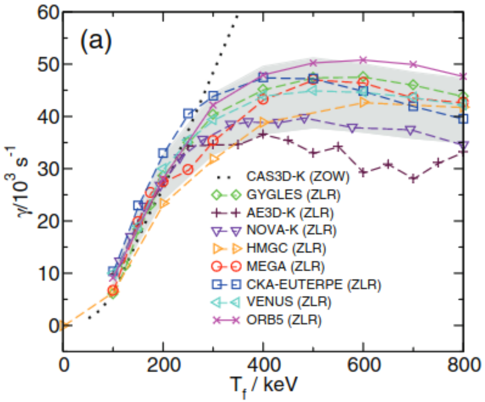
  A circular large aspect ratio tokamak (R_0=10m, a=1m).

  Ad-hoc equilibrium with B_{center} = 3T.

  A **TAE** mode with (m= 10,11, n= - 6).

  Maxwellian **energetic deuterons** (0-800 keV).





[a] Könies et al. (2018) "Benchmark of gyrokinetic, kinetic MHD and gyrofluid codes for the linear calculation of fast particle driven TAE dynamics"

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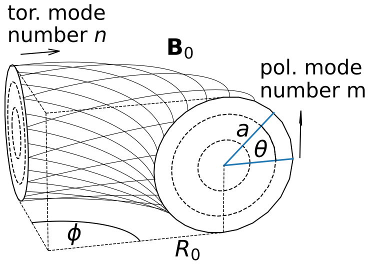

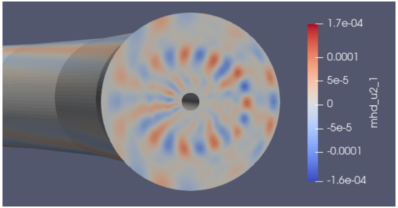

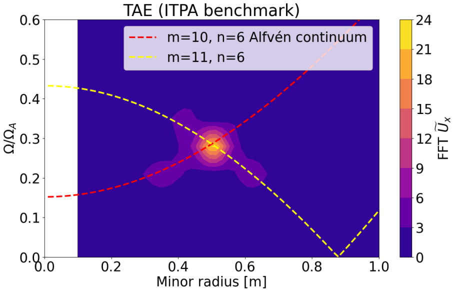

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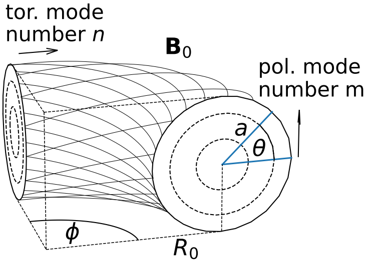

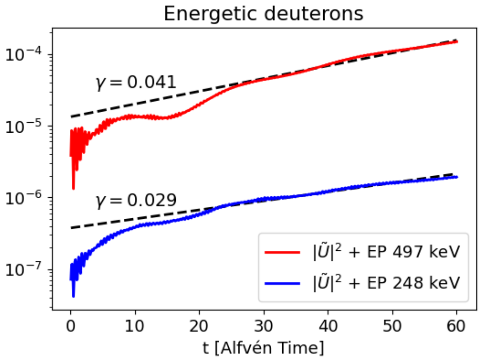

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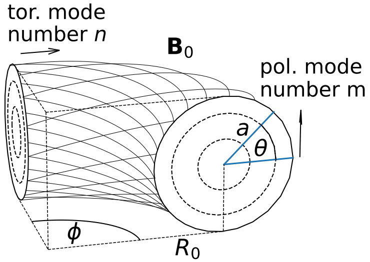

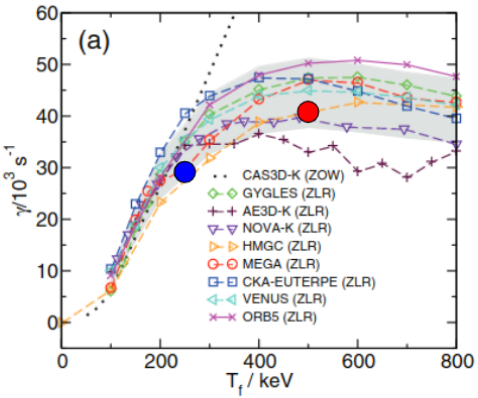

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STRUPHY overview

- **Ongoing projects**
  - Canonical momentum based hybrid kinetic ions massless electrons model.
  - Delta-f Vlasov-Maxwell models.
  - Full MHD model with advanced preconditioning.
  - Few master projects ...

- **Progress in documentation**
  - Easy to learn https://struphy.pages.mpcdf.de/struphy/
  - 9 Tutorials https://struphy.pages.mpcdf.de/struphy/sections/tutorials.html

- **Performance is getting better and better (MPI + OpenMP)**
  - Parallelization for FEEC https://github.com/pyccel/psydac
  - Parallelization for PIC https://github.com/pyccel/pyccel