MAX-PLANCK-INSTITUT FÜR PLASMAPHYSIK

A projection-based approach to handle polar singularities with tensor-product splines

TSVV 10 meeting

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Outline

Motivation

“logical” vs “physical” field spaces

Projection-based approach: a different perspective

Characterizing the pre-polar spline spaces

Computing the conforming projections

Numerical validation in Psydac

Summary

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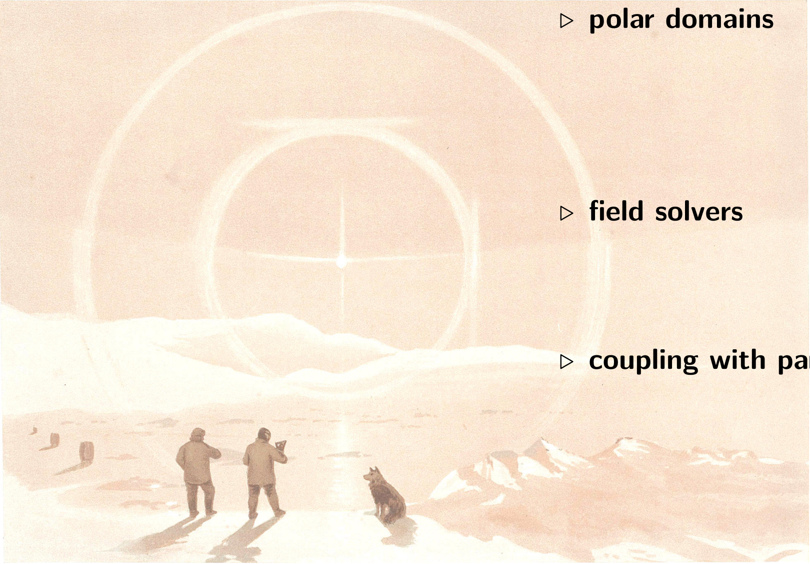
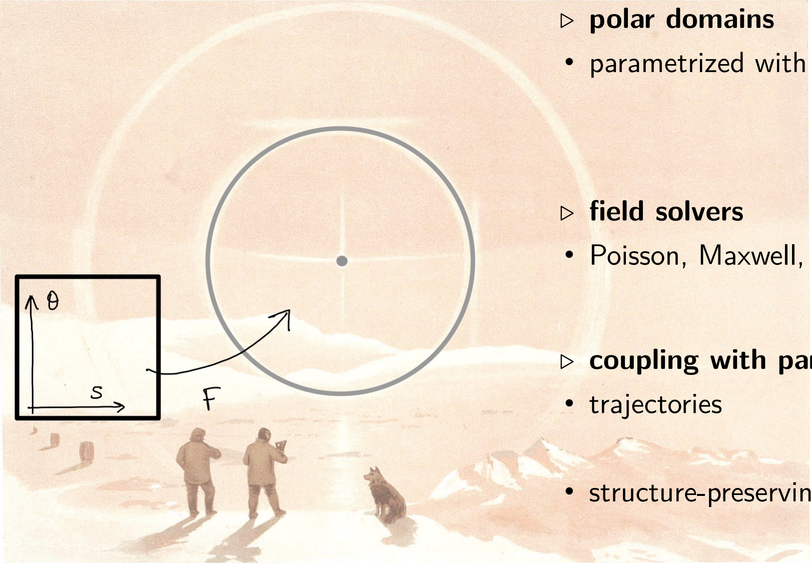
Edward L. Moss “Shores of the Polar Sea. A Narrative of the Arctic Expedition of 1875-6”

Motivation



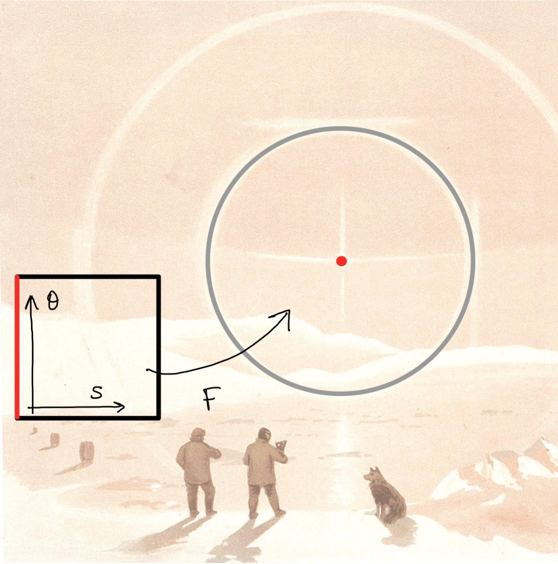
- polar domains
- field solvers
- coupling with particles

Motivation



- ▷ **polar domains**
  - parametrized with tensor-product splines
- ▷ **field solvers**
  - Poisson, Maxwell, curl-curl ...
- ▷ **coupling with particles**
  - trajectories
  - structure-preserving coupling

Motivation



▷ **polar domains**

- parametrized with tensor-product splines
- polar singularity
- ⇝ lack of smoothness (and integrability!)

▷ **field solvers**

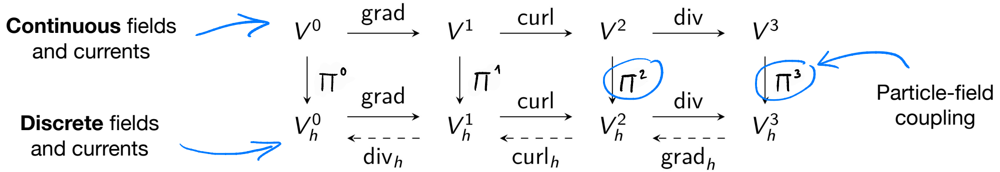
- Poisson, Maxwell, curl-curl ...
- need C^0 potentials, $H(\mathrm{curl})$ fields

▷ **coupling with particles**

- trajectories
- need smoother potentials and fields
- structure-preserving coupling
- need commuting projections

A word about particles

- Variational particle-field discretization in generic commuting de Rham complexes



▷ Action Principle with discrete Lagrangian $\mathcal{L}_h(\boldsymbol{X}_N, \boldsymbol{X}'_N, \boldsymbol{V}_N, \boldsymbol{A}_h, \boldsymbol{A}'_h, \phi_h)$

▷ gauge-free FEM-PIC scheme with

$$\begin{cases} \boldsymbol{E}_h = -\,\mathrm{grad}_h\,\phi_h - \partial_t \boldsymbol{A}_h & (\text{ in } V_h^2\ ) \\ \boldsymbol{B}_h = \mathrm{curl}_h\,\boldsymbol{A}_h & (\text{ in } V_h^1\ ) \end{cases}$$

▷ Hamiltonian structure[1]: energy stability, discrete Casimirs (Gauss laws ...)

[1] Kraus-Kormann-Morrison-Sonnendrücker ('17), CP-Kormann-Sonnendrücker ('21)

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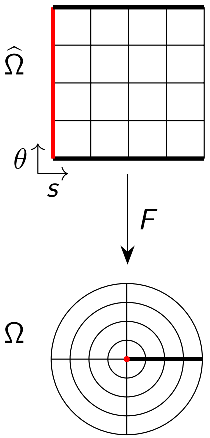
“logical” vs “physical” field spaces

Sergio Leone,“Il buono, il brutto, il cattivo” (1966)

“logical” vs “physical” field spaces: the Good, the Bad and the Ugly



$$
\begin{array}{ccccc}
\widehat{W}_h^0 \ (\subset H^1(\widehat{\Omega})) & \xrightarrow{\text{grad}} & \widehat{W}_h^1 \ (\subset H(\text{curl}; \widehat{\Omega})) & \xrightarrow{\text{curl}} & \widehat{W}_h^2 \ (\subset L^2(\widehat{\Omega})) \\
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- Push-forward operators:

$$
\begin{cases} \mathcal{F}^0 : \hat{\phi} \mapsto \phi, \\ \phi(\boldsymbol{x}) = \hat{\phi}(\hat{\boldsymbol{x}}) \end{cases}
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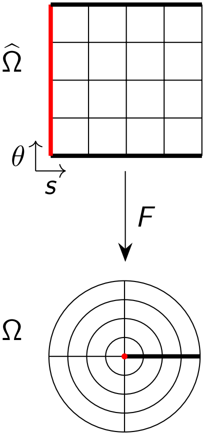
"logical" vs "physical" field spaces: the Good, the Bad and the Ugly



$$\widehat{W}_h^0 \ (\subset H^1(\widehat{\Omega})) \xrightarrow{\text{grad}} \widehat{W}_h^1 \ (\subset H(\text{curl};\widehat{\Omega})) \xrightarrow{\text{curl}} \widehat{W}_h^2 \ (\subset L^2(\widehat{\Omega}))$$

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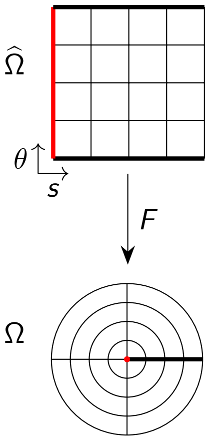

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- with polar map: $DF^{-T} = \begin{pmatrix} \cos\theta & -\frac{1}{s}\sin\theta \\ \sin\theta & \frac{1}{s}\cos\theta \end{pmatrix} \implies \boldsymbol{E}, \beta \notin L^2(\Omega)!$

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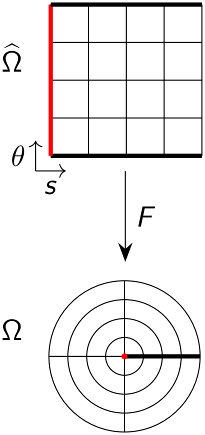

$\widehat{W}_h^0 \ (\subset H^1(\widehat{\Omega})) \xrightarrow{\ \text{grad}\ } \widehat{W}_h^1 \ (\subset H(\text{curl};\widehat{\Omega})) \xrightarrow{\ \text{curl}\ } \widehat{W}_h^2 \ (\subset L^2(\widehat{\Omega}))$

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W_h^0 (? )) — grad? → W_h^1 (?  ; Ω)) — curl? → W_h^2 (? ))

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(moreover, $\text{curl}\, \mathcal{F}^1 \hat{\boldsymbol{E}} = \mathcal{F}^2(\text{curl}\, \hat{\boldsymbol{E}}) + \alpha\delta_{(0,0)}$ with $\alpha = \int_0^{2\pi} \hat{E}_\theta(0,\theta)\, d\theta$)

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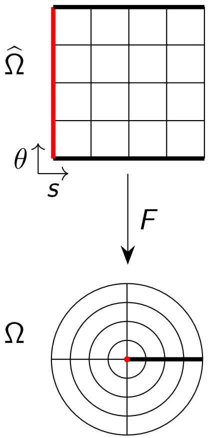

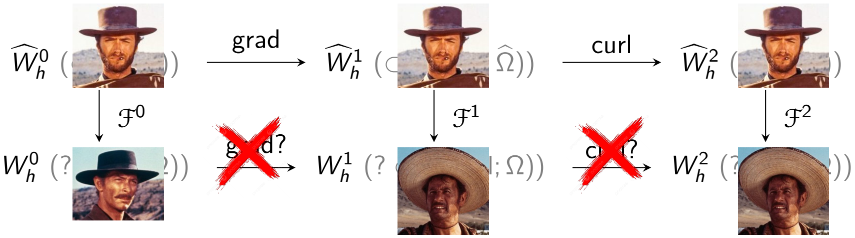

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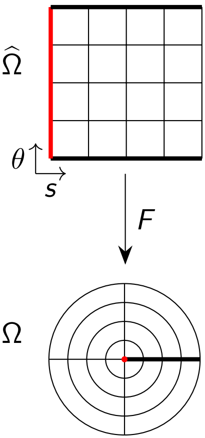

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- Existing solution: use smooth polar splines (non tensor-product) $V^\ell \subset W^\ell$

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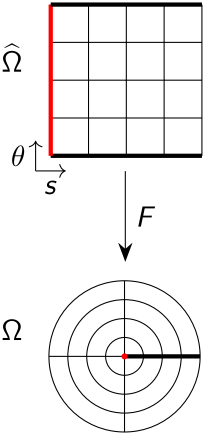

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$$

- Existing solution: use smooth polar splines (non tensor-product) $V^\ell \subset W^\ell$
- Here: use tensor-product splines and project to $\widehat{V}^\ell \subset \widehat{W}^\ell$ s.t. $\mathcal{F}^\ell \widehat{V}^\ell = V^\ell$

Bibliography on polar splines

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- D. Toshniwal & T. J. Hughes *Isogeometric discrete differential forms: Non-uniform degrees, Bézier extraction, polar splines and flows on surfaces*, Computer Methods in Applied Mechanics and Engineering 376 (2021): 113576.
- F. Patrizi *Isogeometric de Rham complex discretization in solid toroidal domains*, arXiv:2106.10470 [math.NA] (2021).
- F. Holderied & S. Possanner *Magneto-hydrodynamic eigenvalue solver for axisymmetric equilibria based on smooth polar splines*, Journal of Computational Physics 464 (2022): 111329.
- A. Bhole, B. Nkonga, S. Pamela, G. Huijsmans, M. Hoelzl & JOREK team *Treatment of polar grid singularities in the bi-cubic Hermite*, Journal of Computational Physics 471 (2022): 111611.

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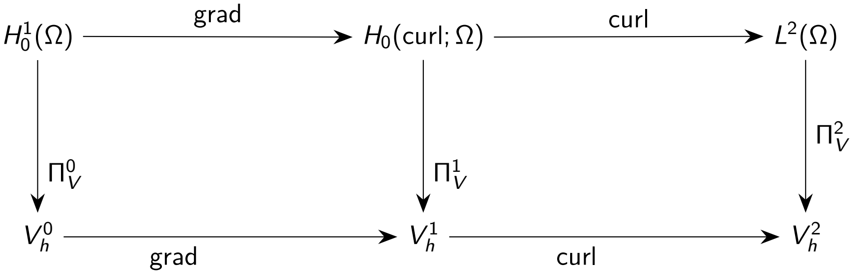
Paolo Uccello,“Battaglia di San Romano” (1438)

Projection-based approach: a different perspective



- Objective #1: commuting de Rham diagram of polar splines

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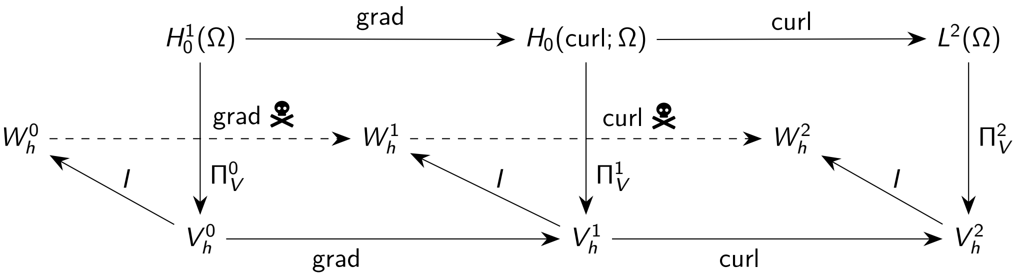
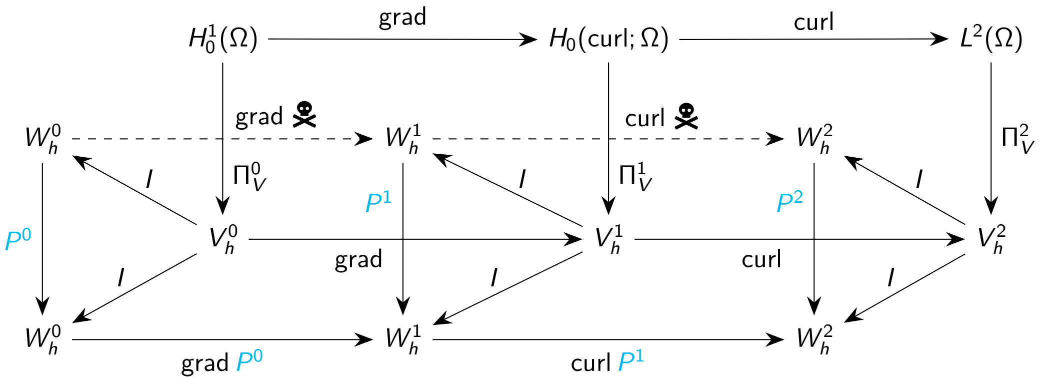


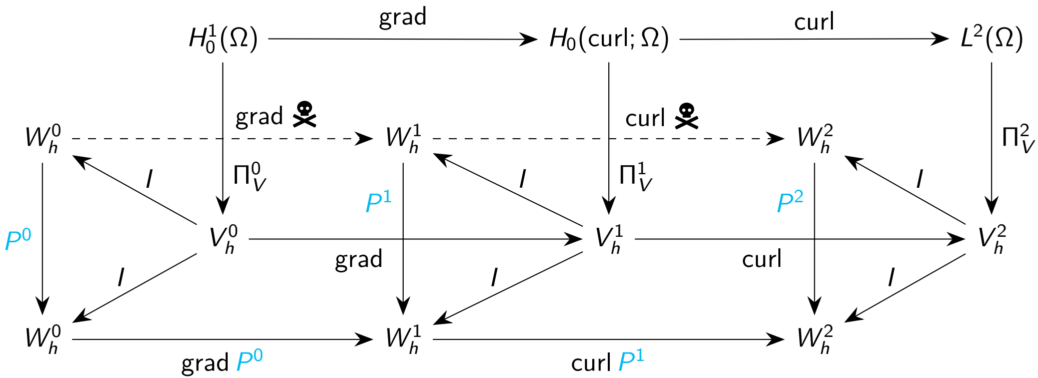
- Objective #1: commuting de Rham diagram of polar splines
- Objective #2: work with tensor-product splines

Projection-based approach: a different perspective



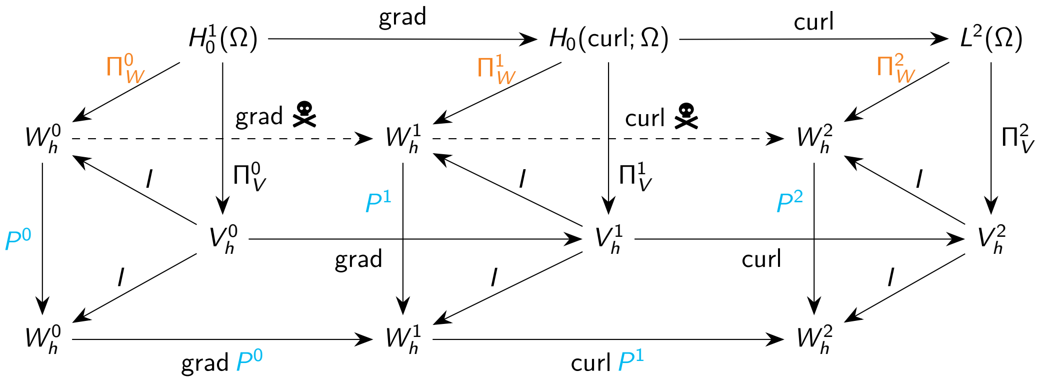
- Objective #1: commuting de Rham diagram of polar splines
- Objective #2: work with tensor-product splines
- Idea: use conforming projections, not conforming bases

Projection-based approach: a different perspective



- Advantage #1: allows re-use of full tensor-product splines spaces and operators
- Advantage #2: similar treatment of multi-patch domains (broken-FEEC)

Projection-based approach: a different perspective



- Advantage #1: allows re-use of full tensor-product splines spaces and operators
- Advantage #2: similar treatment of multi-patch domains (broken-FEEC)
- ⚠ design of good conforming projections → careful design

Conforming vs. projection-based (CONGA) approach: Poisson

- Model and conforming discretization:

$$\begin{cases} -\Delta\phi = f & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \quad \rightsquigarrow \quad \mathbb{S}_p \boldsymbol{\phi}_p = \mathbf{f}_p$$

with $(\mathbb{S}_p)_{a,b} = \int_\Omega \nabla\Lambda_a^{0,p} \cdot \nabla\Lambda_a^{0,p} \,\mathrm{d}x\,\mathrm{d}y$ the stiffness matrix in the polar spline basis.

- CONGA discretization:

$$\left(\alpha(\mathbb{I} - \mathbb{P}^0)^T \mathbb{M}(\mathbb{I} - \mathbb{P}^0) + (\mathbb{P}^0)^T \mathbb{S}\mathbb{P}^0\right) \boldsymbol{\phi} = (\mathbb{P}^0)^T \mathbf{f}$$

with $\alpha > 0$ and

$$\begin{cases} \mathbb{M}^0, \mathbb{S} : \text{the mass and stiffness matrices in the full spline basis.} \\ \mathbb{P}^0 : \text{the projection matrix onto the polar splines, in the full spline basis.} \end{cases}$$

Conforming vs. projection-based (CONGA) approach: Maxwell

- Model and conforming discretization:

$$\begin{cases} \partial_t B + \text{curl}\, \boldsymbol{E} = 0, \\ \frac{1}{c^2}\partial_t \boldsymbol{E} - \textbf{curl}\ B = 0, \end{cases} \quad \rightsquigarrow \quad \begin{cases} \partial_t \mathbf{B}_p + \mathbb{C}_p \mathbf{E}_p = 0 \\ \partial_t \mathbb{M}_p^1 \mathbf{E}_p - \mathbb{C}_p^T \mathbb{M}_p^2 \mathbf{B}_p = 0 \end{cases}$$

with curl and mass matrices in the polar spline basis.

- CONGA discretization:

$$\begin{cases} \partial_t \mathbf{B} + \mathbb{C}\mathbb{P}^1 \mathbf{E} = 0 \\ \partial_t \tilde{\mathbb{M}}^1 \mathbf{E} - (\mathbb{C}\mathbb{P}^1)^T \tilde{\mathbb{M}}^2 \mathbf{B} = 0 \end{cases}$$

with curl and (regularized) mass matrices in the full spline basis

- Note: ⚠ the mass matrices must be regularized because $W_h^1, W_h^2 \not\subset L^2(\Omega)$.

We set: $$\tilde{\mathbb{M}}^\ell := \frac{1}{n_s n_\theta}(\mathbb{I} - \mathbb{P}^\ell)^T(\mathbb{I} - \mathbb{P}^\ell) + (\mathbb{P}^\ell)^T \mathbb{M}^\ell \mathbb{P}^\ell$$

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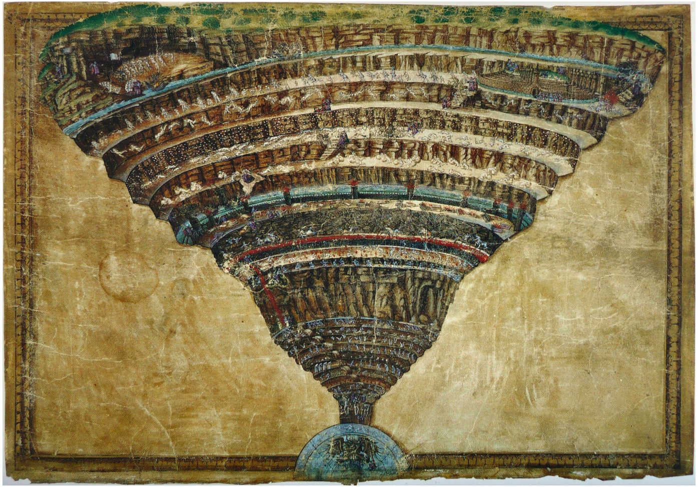
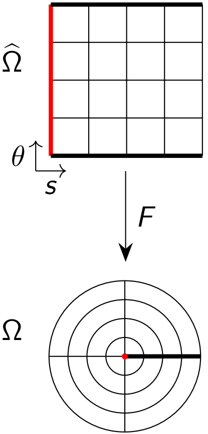


Sandro Botticelli,“Mappa dell’ Inferno” (ca 1490)

Characterizing the pre-polar spline spaces



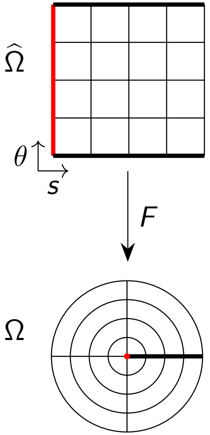
$$F : \begin{pmatrix} s \\ \theta \end{pmatrix} \to \sum_{i,j} \rho_i \begin{pmatrix} \cos\theta_j \\ \sin\theta_j \end{pmatrix} B_i(s)\mathring{B}_k(\theta) \quad \text{with} \quad \begin{cases} \rho_i := \frac{i}{n_s - 1} & 0 \leq i < n_s \\ \theta_j := \frac{2\pi k}{n_\theta} & 0 \leq j < n_\theta \end{cases}$$

$$\begin{cases} \widehat{W}_h^0 = \mathbb{S}^{p_s,p_\theta} = \text{Span}\,\{\widehat{\Lambda}_{ij}^0 := B_i(s)\mathring{B}_j(\theta)\} \\ \widehat{W}_h^1 = \begin{pmatrix} \mathbb{S}^{p_s-1,p_\theta} \\ \mathbb{S}^{p_s,p_\theta-1} \end{pmatrix} = \text{Span}\,\left\{\widehat{\mathbf{\Lambda}}_{ij}^{1,s} = \begin{pmatrix} M_i(s)\mathring{B}_j(\theta) \\ 0 \end{pmatrix}, \widehat{\mathbf{\Lambda}}_{ij}^{1,\theta} = \begin{pmatrix} 0 \\ B_i(s)\mathring{M}_j(\theta) \end{pmatrix}\right\} \\ \widehat{W}_h^2 = \text{Span}\,\{\widehat{\Lambda}_{ij}^2 = M_i(s)\mathring{M}_j(\theta)\} \end{cases}$$

- Pre-polar spline spaces:

(⚠ now $F \in (\mathbb{S}^{p_s,p_\theta})^2$)

Characterizing the pre-polar spline spaces



$$F : \begin{pmatrix} s \\ \theta \end{pmatrix} \to \sum_{i,j} \rho_i \begin{pmatrix} \cos\theta_j \\ \sin\theta_j \end{pmatrix} B_i(s)\mathring{B}_k(\theta) \quad \text{with} \quad \begin{cases} \rho_i := \frac{i}{n_s - 1} & 0 \le i < n_s \\ \theta_j := \frac{2\pi k}{n_\theta} & 0 \le j < n_\theta \end{cases}$$

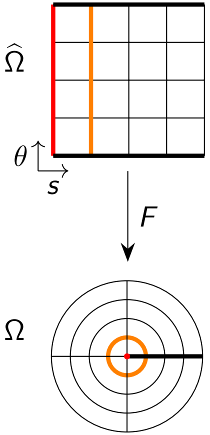
$$\begin{cases} \widehat{W}_h^0 = \mathbb{S}^{p_s,p_\theta} = \text{Span}\left\{\hat{\Lambda}_{ij}^0 := B_i(s)\mathring{B}_j(\theta)\right\} \\ \widehat{W}_h^1 = \begin{pmatrix} \mathbb{S}^{p_s-1,p_\theta} \\ \mathbb{S}^{p_s,p_\theta-1} \end{pmatrix} = \text{Span}\left\{\hat{\mathbf{\Lambda}}_{ij}^{1,s} = \begin{pmatrix} M_i(s)\mathring{B}_j(\theta) \\ 0 \end{pmatrix}, \hat{\mathbf{\Lambda}}_{ij}^{1,\theta} = \begin{pmatrix} 0 \\ B_i(s)\mathring{M}_j(\theta) \end{pmatrix}\right\} \\ \widehat{W}_h^2 = \text{Span}\left\{\hat{\Lambda}_{ij}^2 = M_i(s)\mathring{M}_j(\theta)\right\} \end{cases}$$

- Pre-polar spline spaces: C^0 sequence

$$\begin{cases} \widehat{V}_h^0 = \{\hat{\phi} \in \widehat{W}_h^0 : \mathcal{F}^0\hat{\phi} \in C^0(\Omega)\} \\ \widehat{V}_h^1 = \left\{\hat{\boldsymbol{E}} \in \widehat{W}_h^1 : \mathcal{F}^1\hat{\boldsymbol{E}} \in H(\text{curl};\Omega)\right\} \\ \widehat{V}_h^2 = \{\hat{\beta} \in \widehat{W}_h^2 : \mathcal{F}^2\hat{\beta} \in L^2(\Omega)\} \end{cases}$$

(⚠ now $F \in (\mathbb{S}^{p_s,p_\theta})^2$)

Characterizing the pre-polar spline spaces



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- Pre-polar spline spaces: C^0 sequence

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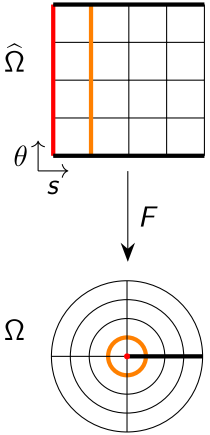
(⚠ now $F \in (\mathbb{S}^{p_s, p_\theta})^2$)

Characterizing the pre-polar spline spaces



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- Pre-polar spline spaces: C^1 sequence

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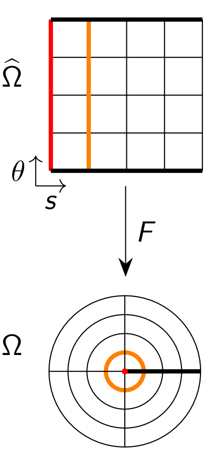
Characterizing the pre-polar spline spaces



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- Pre-polar spline spaces: C^1 sequence

$$\begin{cases} \widehat{U}_h^0 = \{\hat{\phi} \in \widehat{V}_h^0 : \mathcal{F}^0 \hat{\phi} \in C^1(\Omega)\} = \left\{ \hat{\phi} \in \widehat{W}_h^0 : \begin{cases} \phi_{0j} = \phi_{00} \\ \phi_{1j} - \phi_{0j} = \lambda_1 \cos\theta_j + \lambda_2 \sin\theta_j \end{cases} \forall j \right\} \\ \widehat{U}_h^1 = \left\{ \widehat{\boldsymbol{E}} \in \widehat{V}_h^1 : \mathcal{F}^1 \widehat{\boldsymbol{E}} \in C^0(\Omega) \right\} = \left\{ \widehat{\boldsymbol{E}} \in \widehat{W}_h^1 : \begin{cases} E_{0j}^\theta = 0 \\ E_{1j}^\theta = E_{0(j+1)}^s - E_{0j}^s \\ E_{0j}^s = \lambda_1 \cos\theta_j + \lambda_2 \sin\theta_j \end{cases} \forall j \right\} \\ \widehat{U}_h^2 = \{\hat{\beta} \in \widehat{V}_h^2 : \mathcal{F}^2 \hat{\beta} \in L^2(\Omega)\} = \widehat{V}_h^2 \end{cases}$$

(⚠ now $F \in (\mathbb{S}^{p_s,p_\theta})^2$)

Outline

Motivation

"logical" vs "physical" field spaces

Projection-based approach: a different perspective

Characterizing the pre-polar spline spaces

Computing the conforming projections

Numerical validation in Psydac

Summary

Computing the conforming projections

Cimbali, macchina M100 in Firenze (ca 2020)

Determination of the CONGA Projectors

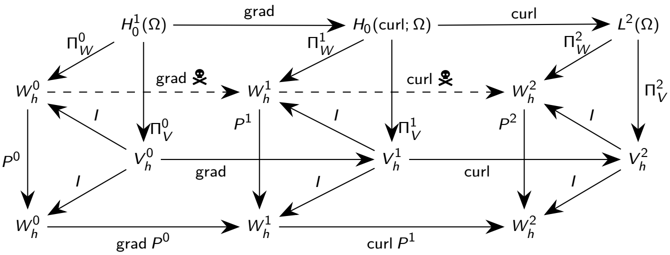
P^ℓ operators on $W_{\boldsymbol{h}}^\ell$, for $\ell = 0, 1, 2$, characterized by:

Determination of the CONGA Projectors

P^ℓ operators on W_h^ℓ, for $\ell = 0, 1, 2$, characterized by:

- Range property: $\mathrm{Im}(P^\ell) \subseteq V_h^\ell$
- Projection property: $P^\ell u = u$ for $u \in V_h^\ell$
- Commuting property: $\begin{cases} \mathrm{grad} \circ P^0 = P^1 \circ \mathrm{grad} & \text{on } \mathrm{Im}(\Pi_W^0) \subseteq W_h^0, \\ \mathrm{curl} \circ P^1 = P^2 \circ \mathrm{curl} & \text{on } \mathrm{Im}(\Pi_W^1) \subseteq W_h^1. \end{cases}$

remind:



CONGA Projector P^0 for the C^0 sequence

$$V_h^0 = \left\{ \phi = \sum_{i,j} \phi_{ij} \Lambda_{ij}^0 \in W_h^0 \ : \ \phi_{0j} = \phi_{00} \quad \forall\, j \right\}$$

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$\Rightarrow \Lambda_{ij}^0 \in V_h^0$ for $i \geq 1$

Projection property: $P^0 \phi = \phi$ for $\phi \in V_h^0 \Rightarrow \boxed{P^0 \Lambda_{ij}^0 = \Lambda_{ij}^0 \ \forall\, i \geq 1}$

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Project Λ_{0j}^0 to any function in V_h^0

Symmetry in $j \Rightarrow$ $\boxed{P^0 \Lambda_{0j}^0 = \frac{1}{n_\theta} \sum_{k=0}^{n_\theta - 1} \Lambda_{0k}^0}$

CONGA Projector P^0 for the C^1 sequence

$\phi \in W_h^0$, $\phi = \mathbf{\Lambda}^{0,T}\boldsymbol{\phi}$, we set

$$P^0\phi = \mathbf{\Lambda}^{0,T}(\mathbb{P}^0\boldsymbol{\phi}) \qquad \text{(square matrix)}$$

with $\mathbb{P}^0 = \mathbb{E}^{0,T}\mathbb{S}^0$, $\mathbb{S}^0$ to be defined.

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with $\boldsymbol{\varphi} = \mathbb{S}^0\boldsymbol{\phi}$.In order for P^0 to have the Projection property we look for $\mathbb{S}^0$ such that $\mathbb{S}^0\mathbb{E}^{0,T} = \mathbb{I}$ (i.e., $\mathbb{S}^0$ left-inverse $\mathbb{E}^{0,T}$).

Thereby, by recalling that $V_h^0 \ni \varphi = \boldsymbol{S}^{0,T}\boldsymbol{\varphi} = \mathbf{\Lambda}^{0,T}(\mathbb{E}^{0,T}\boldsymbol{\varphi})$, we get the Projection property

$$P^0\varphi = \mathbf{\Lambda}^{0,T}(\mathbb{P}^0\mathbb{E}^{0,T}\boldsymbol{\varphi}) = \mathbf{\Lambda}^{0,T}(\mathbb{E}^{0,T}\underbrace{\mathbb{S}^0\mathbb{E}^{0,T}}_{\mathbb{I}}\boldsymbol{\varphi}) = \mathbf{\Lambda}^{0,T}(\mathbb{E}^{0,T}\boldsymbol{\varphi}) = \boldsymbol{S}^{0,T}\boldsymbol{\varphi} = \varphi.$$

CONGA Projector P^0 for the C^1 sequence

$\phi \in W_h^0$, $\phi = \mathbf{\Lambda}^{0,T}\boldsymbol{\phi}$, we set

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Left-inverse of a rectangular matrix not unique.

$$\mathbb{E}^0 = \begin{bmatrix} \mathbb{X}^0 & 0 \\ 0 & \mathbb{I} \end{bmatrix} \quad \Rightarrow \quad \mathbb{S}^0 = \begin{bmatrix} \boldsymbol{s}^0 & 0 \\ 0 & \mathbb{I} \end{bmatrix}, \qquad \text{with } \boldsymbol{s}^0 = \boldsymbol{s}^0_\gamma \text{ determined from symmetry arguments}$$

CONGA Projectors P^1, P^2 for the C^0 sequence

$$V_h^1 = \left\{ \boldsymbol{\psi} = \sum_{i,j} \psi^s_{i,j} \Lambda^{1,s}_{ij} + \sum_{i,j} \psi^\theta_{i,j} \Lambda^{1,\theta}_{ij} \in W_h^1 \ : \ \psi^\theta_{0j} = 0 \wedge \psi^\theta_{1j} = \psi^s_{0(j+1)} - \psi^s_{0j} \ \forall\, j \right\}$$

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$\Rightarrow \Lambda_{ij}^{1,s}$ for $i \geq 1$ and $\Lambda_{ij}^{1,\theta}$ for $i \geq 2$ are in V_h^1

Projection property: $P^1 \boldsymbol{\psi} = \boldsymbol{\psi}$ for $\boldsymbol{\psi} \in V_h^1 \Rightarrow$ $\boxed{P^1 \Lambda_{ij}^{1,s} = \Lambda_{ij}^{1,s} \ \forall\, i \geq 1 \wedge P^1 \Lambda_{ij}^{1,\theta} = \Lambda_{ij}^{1,\theta} \ \forall\, i \geq 2}$

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$\Lambda_{0j}^{1,s}, \Lambda_{0j}^{1,\theta}, \Lambda_{1j}^{1,\theta} \notin V_h^1$ BUT $\boldsymbol{\psi}_j = \Lambda_{0j}^{1,s} + \Lambda_{1(j-1)}^{1,\theta} - \Lambda_{1j}^{1,\theta} \in V_h^1$

$$\Rightarrow P^1 (\Lambda_{0j}^{1,s} + \Lambda_{1(j-1)}^{1,\theta} - \Lambda_{1j}^{1,\theta}) = \Lambda_{0j}^{1,s} + \Lambda_{1(j-1)}^{1,\theta} - \Lambda_{1j}^{1,\theta}.$$

CONGA Projectors P^1, P^2 for the C^0 sequence

$$\Rightarrow \begin{cases} P^1\Lambda^{1,\theta}_{0j} = \boldsymbol{Q}_{0j} \\ P^1\Lambda^{1,\theta}_{1j} = \boldsymbol{Q}_{1j} \\ P^1\Lambda^{1,s}_{0j} = \Lambda^{1,s}_{0j} + \Lambda^{1,\theta}_{1(j-1)} - \Lambda^{1,\theta}_{1j} + \boldsymbol{Q}_{1j} - \boldsymbol{Q}_{1(j-1)} \end{cases}$$

with $\boldsymbol{Q}_{0j}, \boldsymbol{Q}_{1j} \in V^1_h$ to be determined by imposing commutation with P^2.

Remark: Commutation with P^0 true for any choice of P^1, as long as it is a projector, as one can show that $\mathrm{Im}\Pi^0_W \subseteq V^0_h$, so that $\mathrm{grad}\, P^0\phi = \mathrm{grad}\,\phi = P^1\,\mathrm{grad}\,\phi$.

CONGA Projectors P^1, P^2 for the C^0 sequence

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CONGA Projectors P^1, P^2 for the C^0 sequence

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CONGA Projectors P^1, P^2 for the C^0 sequence

Proposition: if $G_j = \operatorname{curl} \boldsymbol{Q}_{1j} + \Lambda^2_{1j}$ then $\operatorname{curl} P^1 \boldsymbol{\psi} = P^2 \operatorname{curl} \boldsymbol{\psi} \quad \forall \boldsymbol{\psi} \in \operatorname{Im}\Pi^1_W$.

$$\Rightarrow \begin{cases} P^1 \Lambda^{1,\theta}_{0j} = \boldsymbol{Q}_{0j} \\ P^1 \Lambda^{1,\theta}_{1j} = \boldsymbol{Q}_{1j} \\ P^1 \Lambda^{1,s}_{0j} = \Lambda^{1,s}_{0j} + \Lambda^{1,\theta}_{1(j-1)} - \Lambda^{1,\theta}_{1j} + \boldsymbol{Q}_{1j} - \boldsymbol{Q}_{1(j-1)} \\ P^1 \Lambda^{1,s}_{ij} = \Lambda^{1,s}_{ij} \quad \forall\, i \geq 1 \\ P^1 \Lambda^{1,\theta}_{ij} = \Lambda^{1,\theta}_{ij} \quad \forall\, i \geq 2 \end{cases} \qquad \begin{cases} P^2 \Lambda^2_{0j} = G_j \\ P^2 \Lambda^2_{ij} = \Lambda^2_{ij} \quad \forall\, i \geq 1. \end{cases}$$

CONGA Projectors P^1, P^2 for the C^0 sequence

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P^1 \Lambda^{1,s}_{ij} = \Lambda^{1,s}_{ij} \quad \forall i \geq 1 \\
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CONGA Projectors P^1, P^2 for the C^0 sequence

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This choice is convenient as all the projections are local and invariant w.r.t. j.

CONGA Projectors P^1, P^2 for the C^1 sequence

$$V_h^1 = \left\{ \psi = \mathbf{\Lambda}^{1,T}\boldsymbol{\psi} = \begin{pmatrix} (\mathbf{\Lambda}^{1,s})^T\boldsymbol{\psi}^s \\ (\mathbf{\Lambda}^{1,\theta})^T\boldsymbol{\psi}^\theta \end{pmatrix} \;:\; \begin{aligned} &\psi^s_{0j} = \lambda_1 \cos\theta_j + \lambda_2 \sin\theta_j \\ &\psi^\theta_{0j} = 0 \\ &\psi^\theta_{1j} = \psi^s_{0(j+1)} - \psi^s_{0j} \end{aligned} \right\} \text{ with } \mathbf{\Lambda}^1 = \begin{pmatrix} \mathbf{\Lambda}^{1,s} \\ \mathbf{\Lambda}^{1,\theta} \end{pmatrix}, \boldsymbol{\psi} = \begin{pmatrix} \boldsymbol{\psi}^s \\ \boldsymbol{\psi}^\theta \end{pmatrix}$$

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Initial guess for P^1 by looking at V_h^1 characterization and local exactness

CONGA Projectors P^1, P^2 for the C^1 sequence

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$$\begin{cases} P^1 \Lambda_{0j}^{1,s} = \text{grad } P^0 \Lambda_{1j}^0 + \boldsymbol{R}_j & P^1 \Lambda_{0j}^{1,\theta} = \boldsymbol{Q}_{0j}, \\ P^1 \Lambda_{ij}^{1,s} = \Lambda_{ij}^{1,s} \text{ for } i \geq 1 & P^1 \Lambda_{1j}^{1,\theta} = \boldsymbol{Q}_{1j}, \\ P^1 \Lambda_{ij}^{1,\theta} = \Lambda_{ij}^{1,\theta} \text{ for } i \geq 2. & \end{cases}$$

CONGA Projectors P^1, P^2 for the C^1 sequence

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CONGA Projectors P^1, P^2 for the C^1 sequence

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CONGA Projectors P^1, P^2 for the C^1 sequence

$$V_h^2 = \{\eta = \mathbf{\Lambda}^{2,T}\boldsymbol{\eta} \, : \, \eta_{0j} = 0\}$$

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☹

CONGA Projectors P^1, P^2 for the C^1 sequence

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$$P^2\Lambda_{ij}^2 = \begin{cases} \Lambda_{1j}^2 & i = 0 \\ \Lambda_{ij}^2 & i \geq 1 \end{cases} \xrightarrow{\text{commuting property}} \begin{cases} \boxed{\text{curl } \boldsymbol{R}_j = \text{curl } \Lambda_{1j}^{1,s}} \\ \boldsymbol{Q}_{1(j-1)} - \boldsymbol{Q}_{1j} = \Lambda_{1j}^{1,s} - \boldsymbol{R}_j \\ \sum_{j=0}^{n_\theta - 1} \boldsymbol{R}_j = \sum_{j=0}^{n_\theta - 1} \Lambda_{1j}^{1,s} \end{cases}$$

CONGA Projectors P^1, P^2 for the C^1 sequence

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$$P^2\Lambda^2_{ij} = \begin{cases} \Lambda^2_{1j} & i = 0 \\ \Lambda^2_{ij} & i \geq 1 \end{cases} \xrightarrow{\text{commuting property}} \begin{cases} \boxed{\text{curl } \boldsymbol{R}_j = \text{curl } \Lambda^{1,s}_{1j}} \\ \boldsymbol{Q}_{1(j-1)} - \boldsymbol{Q}_{1j} = \Lambda^{1,s}_{1j} - \boldsymbol{R}_j \\ \sum_{j=0}^{n_\theta - 1} \boldsymbol{R}_j = \sum_{j=0}^{n_\theta - 1} \Lambda^{1,s}_{1j} \end{cases} \Rightarrow P^1 \text{ local} \quad 😄$$

CONGA Projectors P^1, P^2 for the C^1 sequence

We still have to determine $P^1 \dots$

Projector P^1

$$\begin{cases} P^1 \Lambda^{1,s}_{0j} = \text{grad } P^0 \Lambda^0_{1j} + \boldsymbol{R}_j & P^1 \Lambda^{1,\theta}_{0j} = \boldsymbol{Q}_{0j}, \\ P^1 \Lambda^{1,s}_{ij} = \Lambda^{1,s}_{ij} \text{ for } i \geq 1 & P^1 \Lambda^{1,\theta}_{1j} = \boldsymbol{Q}_{1j}, \\ P^1 \Lambda^{1,\theta}_{ij} = \Lambda^{1,\theta}_{ij} \text{ for } i \geq 2. \end{cases}$$

Constraints

$$\begin{cases} \text{curl } \boldsymbol{R}_j = \text{curl } \Lambda^{1,s}_{1j} \\ \boldsymbol{Q}_{1(j-1)} - \boldsymbol{Q}_{1j} = \Lambda^{1,s}_{1j} - \boldsymbol{R}_j \\ \sum_{j=0}^{n_\theta - 1} \boldsymbol{R}_j = \sum_{j=0}^{n_\theta - 1} \Lambda^{1,s}_{1j} \end{cases}$$

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CONGA Projectors P^1, P^2 for the C^1 sequence

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Constraints

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CONGA Projectors P^1, P^2 for the C^1 sequence

We still have to determine P^1 ...

Projector P^1

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Constraints

$$\begin{cases} \text{curl } \boldsymbol{R}_j = \text{curl } \Lambda^{1,s}_{1j} \\ \boldsymbol{Q}_{1(j-1)} - \boldsymbol{Q}_{1j} = \Lambda^{1,s}_{1j} - \boldsymbol{R}_j \\ \sum_{j=0}^{n_\theta - 1} \boldsymbol{R}_j = \sum_{j=0}^{n_\theta - 1} \Lambda^{1,s}_{1j} \end{cases}$$

CONGA Projectors P^1, P^2 for the C^1 sequence

We still have to determine $P^1 \dots$

Projector P^1

$$
\begin{cases}
P^1\Lambda^{1,s}_{0j} = \text{grad } P^0\Lambda^0_{1j} + \boldsymbol{R}_j & P^1\Lambda^{1,\theta}_{0j} = \boldsymbol{0}, \\
P^1\Lambda^{1,s}_{ij} = \Lambda^{1,s}_{ij} \text{ for } i \geq 1 & P^1\Lambda^{1,\theta}_{1j} = \boldsymbol{0}, \\
P^1\Lambda^{1,\theta}_{ij} = \Lambda^{1,\theta}_{ij} \text{ for } i \geq 2. &
\end{cases}
$$

Constraints

$$
\begin{cases}
\text{curl } \boldsymbol{R}_j = \text{curl } \Lambda^{1,s}_{1j} \\
\boldsymbol{Q}_{1(j-1)} - \boldsymbol{Q}_j = \Lambda^{1,s}_{1j} - \boldsymbol{R}_j \\
\displaystyle\sum_{j=0}^{n_\theta - 1} \boldsymbol{R}_j = \sum_{j=0}^{n_\theta - 1} \Lambda^{1,s}_{1j}
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$$

CONGA Projectors P^1, P^2 for the C^1 sequence

We still have to determine $P^1 \ldots$

Projector P^1

$$\begin{cases} P^1\Lambda^{1,s}_{0j} = \text{grad } P^0\Lambda^0_{1j} + \boldsymbol{R}_j & P^1\Lambda^{1,\theta}_{0j} = \boldsymbol{0}, \\ P^1\Lambda^{1,s}_{ij} = \Lambda^{1,s}_{ij} \text{ for } i \geq 1 & P^1\Lambda^{1,\theta}_{1j} = \boldsymbol{0}, \\ P^1\Lambda^{1,\theta}_{ij} = \Lambda^{1,\theta}_{ij} \text{ for } i \geq 2. & \end{cases}$$

Constraints

$$\begin{cases} \text{curl } \boldsymbol{R}_j = \text{curl } \Lambda^{1,s}_{1j} \\ \boldsymbol{Q}_{1(j-1)} - \boldsymbol{Q}_{1j} = \Lambda^{1,s}_{1j} - \boldsymbol{R}_j \\ \sum_{j=0}^{n_\theta - 1} \boldsymbol{R}_j = \sum_{j=0}^{n_\theta - 1} \Lambda^{1,s}_{1j} \end{cases}$$

CONGA Projectors P^1, P^2 for the C^1 sequence

We still have to determine $P^1 \dots$

Projector P^1

$$\begin{cases} P^1\Lambda^{1,s}_{0j} = \text{grad } P^0\Lambda^0_{1j} + \boldsymbol{R}_j & P^1\Lambda^{1,\theta}_{0j} = \mathbf{0}, \\ P^1\Lambda^{1,s}_{ij} = \Lambda^{1,s}_{ij} \text{ for } i \geq 1 & P^1\Lambda^{1,\theta}_{1j} = \mathbf{0}, \\ P^1\Lambda^{1,\theta}_{ij} = \Lambda^{1,\theta}_{ij} \text{ for } i \geq 2. & \end{cases}$$

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$$\begin{cases} \text{curl } \boldsymbol{R}_j = \text{curl } \Lambda^{1,s}_{1j} \\ \mathbf{0} = \Lambda^{1,s}_{1j} - \boldsymbol{R}_j \\ \sum_{j=0}^{n_\theta - 1} \boldsymbol{R}_j = \sum_{j=0}^{n_\theta - 1} \Lambda^{1,s}_{1j} \end{cases}$$

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Constraints

$$\begin{cases} \text{curl } R_j = \text{curl } \Lambda^{1,s}_{1j} \\ \mathbf{0} = \Lambda^{1,s}_{1j} - \boldsymbol{R}_j \\ \sum_{j=0}^{n_\theta-1} \boldsymbol{R}_j = \sum_{j=0}^{n_\theta-1} \Lambda^{1,s}_{1j} \end{cases}$$

C^0 CONGA Projectors: Matrix Form

$\mathbb{P}^0, \mathbb{P}^1, \mathbb{P}^2$ sparse matrices with the following structure:

$$\mathbb{P}^0 = \begin{bmatrix} \frac{1}{n_\theta}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix}$$

$$\mathbb{P}^1 = \begin{bmatrix} \multicolumn{2}{c}{\boldsymbol{I}} & \multicolumn{2}{c}{\mathbf{0}} \\ \begin{matrix}\mathbf{0} \\ \mathring{\boldsymbol{d}}\end{matrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{I} \end{bmatrix}$$

$$\mathbb{P}^2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \begin{matrix}\boldsymbol{I} \\ \mathbf{0}\end{matrix} & \boldsymbol{I} \end{bmatrix}$$

where $\mathring{\boldsymbol{d}} = \begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ 1 & & & -1 \end{bmatrix}$ and $\mathbf{1}$ is the matrix full of 1's (both of size $n_\theta \times n_\theta$)

C^1 CONGA Projectors: Matrix Form

$\mathbb{P}^0, \mathbb{P}^1, \mathbb{P}^2$ sparse matrices with the following structure:

$$\mathbb{P}^0 = \left(\begin{array}{cc|c} \frac{1}{n_\theta}\mathbf{1} & \mathbf{0} & \\ \frac{1}{n_\theta}\mathbf{1} & \boldsymbol{p} & \mathbf{0} \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \boldsymbol{I} \end{array}\right)$$

$$\mathbb{P}^1 = \left(\begin{array}{c|c|c|c} \boldsymbol{p} & \mathbf{0} & \multicolumn{2}{c}{} \\ \boldsymbol{I}-\boldsymbol{p} & & \multicolumn{2}{c}{\mathbf{0}} \\ \mathbf{0} & \boldsymbol{I} & \multicolumn{2}{c}{} \\ \hline \mathbf{0} & & & \\ \mathring{\boldsymbol{d}}\boldsymbol{p} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{I} \end{array}\right)$$

$$\mathbb{P}^2 = \left(\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \boldsymbol{I} & \\ \mathbf{0} & \boldsymbol{I} \end{array}\right)$$

where $\boldsymbol{p}$ is a Toeplitz matrix with $\boldsymbol{p}_{\ell,k} = \frac{2}{n_\theta}\cos(\theta_\ell - \theta_k)$ for $\ell, k = 0, \ldots, n_\theta - 1$

Outline

Motivation

"logical" vs "physical" field spaces

Projection-based approach: a different perspective

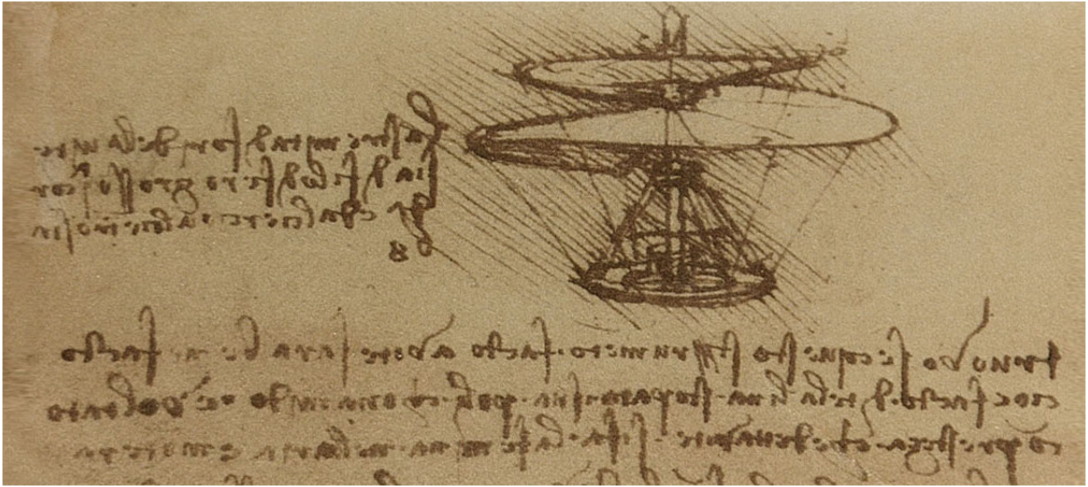
Characterizing the pre-polar spline spaces

Computing the conforming projections

Numerical validation in Psydac

Summary

Numerical validation in Psydac



Leonardo da Vinci, prototype design (ca 1490)

Reminder: CONGA (projection-based) Poisson solver

- Model and conforming discretization:

$$\begin{cases} -\Delta\phi = f & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \qquad \rightsquigarrow \qquad \mathbb{S}_p \boldsymbol{\phi}_p = \mathbf{f}_p$$

with $(\mathbb{S}_p)_{a,b} = \int_\Omega \nabla\Lambda_a^{0,p} \cdot \nabla\Lambda_a^{0,p} \, dx \, dy$ the stiffness matrix in the polar spline basis.

- CONGA discretization:

$$\Big(\alpha(\mathbb{I} - \mathbb{P}^0)^T \mathbb{M}(\mathbb{I} - \mathbb{P}^0) + (\mathbb{P}^0)^T \mathbb{S}\mathbb{P}^0\Big) \boldsymbol{\phi} = (\mathbb{P}^0)^T \mathbf{f}$$

with $\alpha > 0$ and

$$\begin{cases} \mathbb{M}^0, \mathbb{S} : \text{the mass and stiffness matrices in the full spline basis.} \\ \mathbb{P}^0 : \text{the projection matrix onto the polar splines, in the full spline basis.} \end{cases}$$

Reminder: CONGA (projection-based) Maxwell solver

- Model and conforming discretization:

$$\begin{cases} \partial_t B + \mathrm{curl}\ \boldsymbol{E} = 0, \\ \frac{1}{c^2}\partial_t \boldsymbol{E} - \mathbf{curl}\ B = 0, \end{cases} \quad \rightsquigarrow \quad \begin{cases} \partial_t \mathbf{B}_p + \mathbb{C}_p \mathbf{E}_p = 0 \\ \partial_t \mathbb{M}^1_p \mathbf{E}_p - \mathbb{C}_p^T \mathbb{M}^2_p \mathbf{B}_p = 0 \end{cases}$$

with curl and mass matrices in the polar spline basis.

- CONGA discretization:

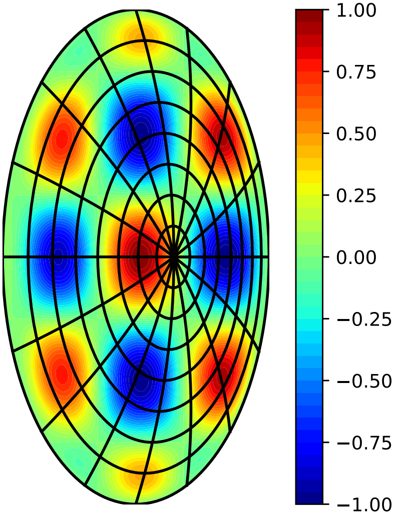
$$\begin{cases} \partial_t \mathbf{B} + \mathbb{C}\mathbb{P}^1 \mathbf{E} = 0 \\ \partial_t \tilde{\mathbb{M}}^1 \mathbf{E} - (\mathbb{C}\mathbb{P}^1)^T \tilde{\mathbb{M}}^2 \mathbf{B} = 0 \end{cases}$$

with curl and (regularized) mass matrices in the full spline basis

- Note: ⚠ the mass matrices must be regularized because $W_h^1, W_h^2 \not\subset L^2(\Omega)$.

We set: $$\tilde{\mathbb{M}}^\ell := \frac{1}{n_s n_\theta}(\mathbb{I} - \mathbb{P}^\ell)^T(\mathbb{I} - \mathbb{P}^\ell) + (\mathbb{P}^\ell)^T \mathbb{M}^\ell \mathbb{P}^\ell$$

Poisson Problem on a Polar Domain

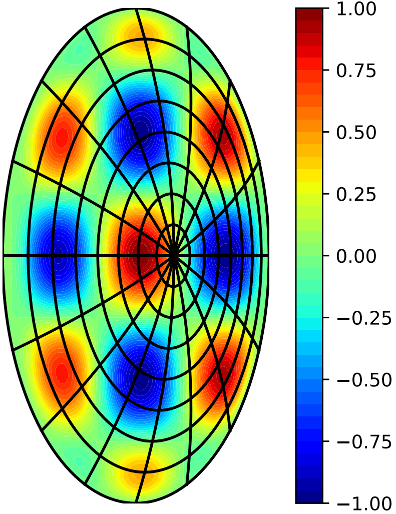


Homogeneous Poisson problem:

$$\begin{cases} -\Delta\phi = f & \Omega \\ \phi = 0 & \partial\Omega \end{cases}$$

ϕ manufactured solution shown on the left.

Poisson Problem on a Polar Domain



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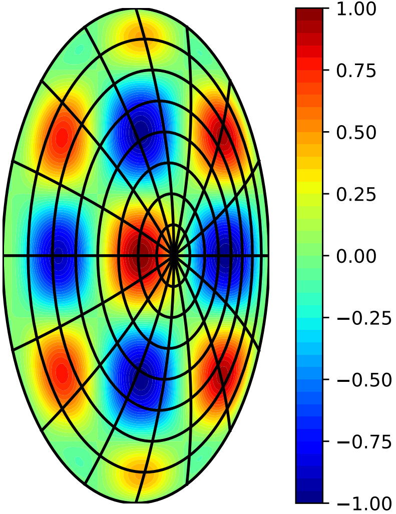
Discretization with degree $p^s, p^\theta = 2, 3, 4$

Poisson Problem on a Polar Domain



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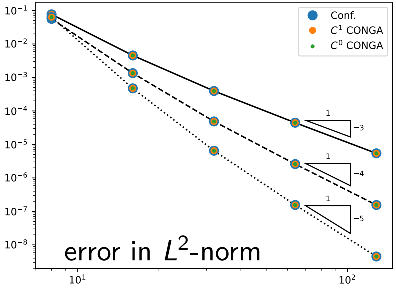
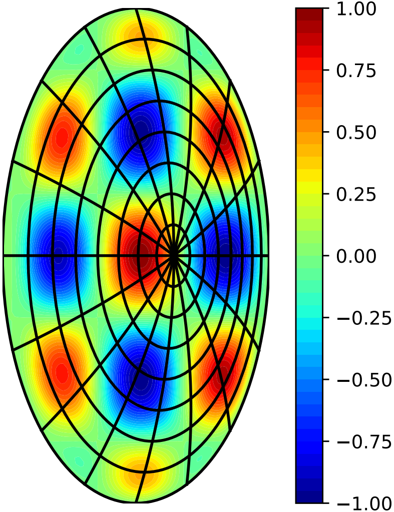


Same (optimal) order of the conforming (polar) discretization

Poisson Problem on a Polar Domain



Homogeneous Poisson problem:

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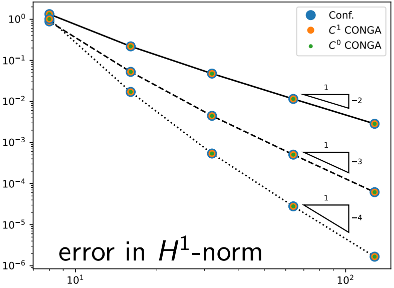
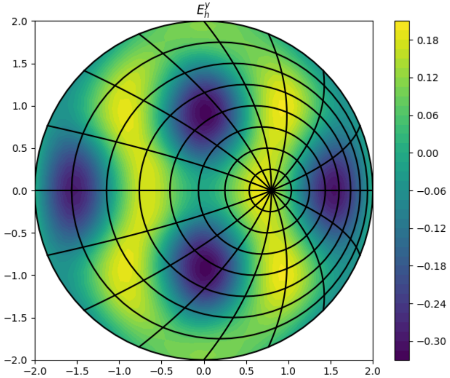


Same (optimal) order of the conforming (polar) discretization

Maxwell Problem on a Polar Domain

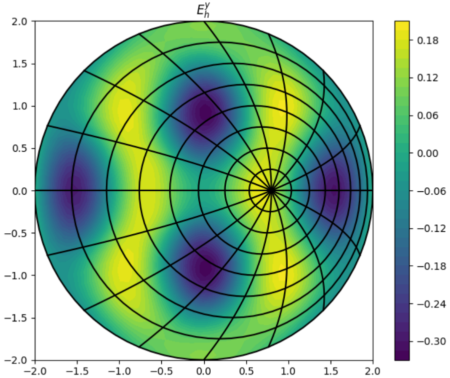


TD Maxwell problem:

$$\begin{cases} \partial_t B + \operatorname{curl} \boldsymbol{E} = 0, \\ \frac{1}{c^2}\partial_t \boldsymbol{E} - \mathbf{curl}\ B = 0, \end{cases}$$

$\boldsymbol{E}, B$: Fourier-Bessel eigenmode, shown on the left.

Maxwell Problem on a Polar Domain



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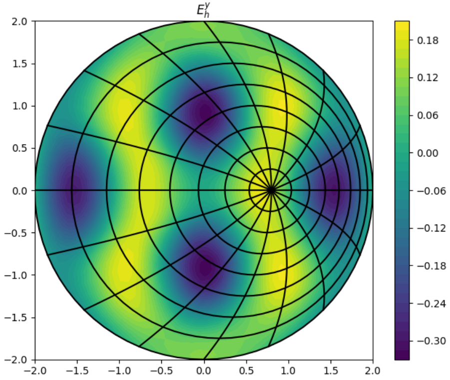
Discretization with degree $p^s, p^\theta = 2, 3, 4$

Maxwell Problem on a Polar Domain



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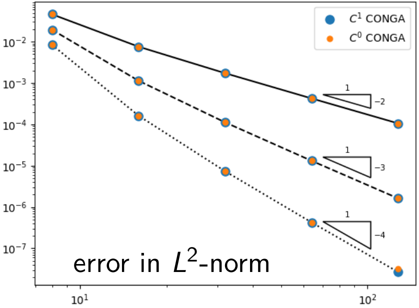
$\boldsymbol{E}, B$: Fourier-Bessel eigenmode, shown on the left.

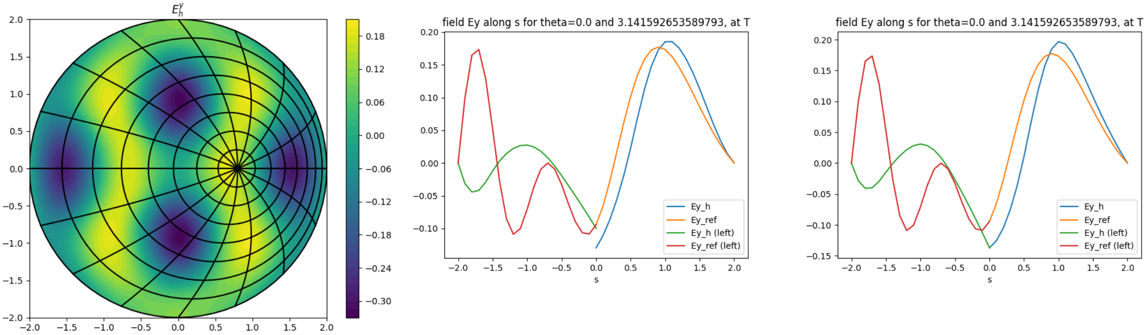
Discretization with degree $p^s, p^\theta = 2, 3, 4$



Optimal order of convergence is observed

Maxwell Problem on a Polar Domain

Cuts along $\theta = 0$ (with C^0 vs. C^1 projections)



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Summary

- Poisson equation: $(\alpha(\mathbb{I}-\mathbb{P}^0)^T\mathbb{M}^0(\mathbb{I}-\mathbb{P}^0)+(\mathbb{P}^0)^T\mathbb{S}\mathbb{P}^0)\boldsymbol{\phi}=(\mathbb{P}^0)^T\mathbf{f}$
- Maxwell equations: $\partial_t\mathbf{B}+\mathbb{C}\mathbb{P}^1\mathbf{E}=0,\ \partial_t\tilde{\mathbb{M}}^1\mathbf{E}-(\mathbb{C}\mathbb{P}^1)^T\tilde{\mathbb{M}}^2\mathbf{B}=0$
- $\mathbb{C}$, $\mathbb{S}$: usual curl and stiffness matrices, $\mathbb{M}^0, \tilde{\mathbb{M}}^1, \tilde{\mathbb{M}}^2$: (regularized) mass matrices
- projection matrices:

$$\mathbb{P}^0=\left[\begin{array}{c|c|c} \frac{1}{n_\theta}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline \frac{1}{n_\theta}\mathbf{1} & \boldsymbol{p} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \boldsymbol{I} \end{array}\right] \qquad \mathbb{P}^1=\left[\begin{array}{c|c|c|c} \boldsymbol{p} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \boldsymbol{I}-\boldsymbol{p} & \boldsymbol{I} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{I} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathring{\boldsymbol{d}}\boldsymbol{p} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{I} \end{array}\right] \qquad \mathbb{P}^2=\left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \boldsymbol{I} & \boldsymbol{I} \\ \hline \mathbf{0} & \boldsymbol{I} \end{array}\right]$$

with $\mathring{\boldsymbol{d}}=\begin{bmatrix}-1 & 1 & \\ & \ddots & \ddots\end{bmatrix}$, $\mathbf{1}_{\ell,k}=1$ and $\boldsymbol{p}_{\ell,k}=\frac{2}{n_\theta}\cos(\theta_\ell-\theta_k)$ for $0\le\ell,k<n_\theta$