Full-f drift-kinetic plus δ-f gyrokinetic turbulent simulations of a linear plasma device

June 20, 2025

J. E. Mencke, P. Ricci

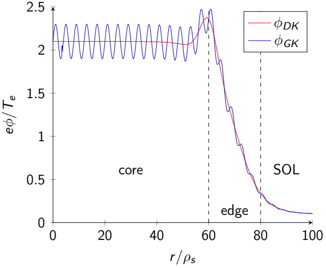
Ecole Polytechnique Fédérale de Lausanne (EPFL), Swiss Plasma Center (SPC), Lausanne, Switzerland

Splitting of fields

$$\phi = \phi_{DK} + \phi_{Gk},$$
$$\phi_{DK} \sim eT_e, \quad \phi_{GK} \sim \epsilon_\delta \phi_{DK},$$
$$\epsilon_\perp \sim \rho_s \left|\nabla_\perp \ln \phi_{DK}\right| < 1,$$
$$\rho_s \left|\nabla_\perp \ln \phi_{GK}\right| \sim 1,$$
$$F_a \sim F_{aDK} + F_{aGK}$$

DK: drift-kinetic ordering (long-wavelength limit)
GK: gyrokinetic ordering (δ-f GK)

Expectations...
Both perturbations



GK δ-f fields are global
No flux-tube approximation

Swiss Plasma Center

Velocity integral:

$$\|f\| = \int_{-\infty}^{\infty} dv_{\|} \int_0^{\infty} d\mu \int_0^{2\pi} d\theta \frac{B}{m_a} f$$

Gyro-average:

$$\langle f \rangle_{\boldsymbol{R}} = \frac{1}{2\pi} \int_0^{2\pi} d\theta f = J_0\left(\frac{v_\perp k_\perp}{\Omega_a}\right) f$$

Adjoint gyro-average:

$$\|\langle f \rangle_{\boldsymbol{R}}\, g\| = \left\| f \, \langle g \rangle_{\boldsymbol{x}}^{\dagger} \right\|, \quad \langle f \rangle_{\boldsymbol{R}} = \langle f \rangle_{\boldsymbol{x}}^{\dagger}$$

$$\tilde{f} = f - \langle f \rangle_{\boldsymbol{R}}$$

For DK variables (e.g. N_i),

$$N_i \sim N_0, \quad N_{i1} \sim \epsilon_\perp^2 N_0,$$

Characteristic frequency

$$\omega \sim \epsilon \epsilon_\perp \Omega_i$$

Parallel wavenumber

$$k_{\|} \sim \epsilon k_\perp$$

Ion-ion collision frequency

$$\nu_{ii} \sim \epsilon_\nu \Omega_i$$

Swiss Plasma Center

Starting from single-particle one-form, deriving Euler-Lagrange equation to find $\dot{\boldsymbol{R}}$, $\dot{\boldsymbol{v}}$, and ϕ_{DK}, ϕ_{GK}

Variation with ϕ_{GK}, require validity for all $\epsilon_\perp$, ϵ_δ

$$\epsilon_\perp^0 \epsilon_\delta^0 : \quad N_i - N_e = 0$$

$$\epsilon_\perp^2 \epsilon_\delta^0 : \quad N_{e1} - N_{i1} = \frac{1}{B\Omega_i} N_i \nabla_\perp^2 \phi_{DK} + \frac{1}{2m_i\Omega_i^2} \nabla_\perp^2 P_{\perp i},$$

$$\epsilon_\perp^0 \epsilon_\delta^1 : \quad \delta\varrho_i^* - \delta N_e = -\frac{q_i}{B} \left\| \partial_\mu \left\langle \left\langle \phi_{GK} \right\rangle_{\boldsymbol{R}} \right\rangle_{\boldsymbol{x}}^\dagger \right\| ,$$

$$\varrho_{iGK}^* = \left\| \left\langle F_{iGK} \right\rangle_{\boldsymbol{R}} / F_{iGK} \right\|_{iGK}$$

(Variation with ϕ_{DK} gives long-wavelength (for $k_{\perp GK}$) limit)

Equation for distribution function through Boltzmann equation

$$F_{aDK}\left(v_{\|},\mu,\boldsymbol{R},t\right)=\sum_{(p,j)=(0,0)}^{(P,J)}\mathcal{N}_{aDK}^{pj}\left(\boldsymbol{R},t\right)H^{pj}\left(s_{a\|},x_a\right)F_{Mi}\left(s_{a\|},x_a\right),$$

$$F_{iGK}\left(v_{\|},\mu,\boldsymbol{R},t\right)=\sum_{(p,j)=(0,0)}^{(P,J)}\mathcal{N}_{GK}^{pj}\left(\boldsymbol{R},t\right)H^{pj}\left(s_{a\|},x_a\right)F_{Mi}\left(s_{a\|},x_a\right),$$

$$H^{pj}\left(s_{a\|},x_a\right)=\frac{H_p\left(s_{a\|}\right)L_j\left(x_a\right)}{\sqrt{2^p p!}},\quad s_{\|}=\frac{v_{\|}-U_{\|a}\left(\boldsymbol{R},t\right)}{v_{Tha0}},$$

$$x_i=\frac{\mu B}{T_{a0}},\quad \int dv H^{pj}H^{kl}F_{Mi}=\delta_k^p\delta_l^j,$$

$$N_{aDK}=\mathcal{N}_{aDK}^{00},\quad \mathcal{N}_{aDK}^{10}=0,\quad \|f\|^{pj}=\left\|H^{pj}f\right\|$$

Higher collisions, fewer moments!

$$\partial_t \left(F_{aDK} + F_{aGK}\right) = ...$$

We define a small scale average

$$\langle g_{aDK} \rangle_{SC} = g_{aDK}, \quad \langle F_{aGK} \rangle_{SC} = 0,$$
$$\langle \nabla_\perp \langle \phi_{GK} \rangle_{\boldsymbol{R}} \rangle_{SC} = \nabla_\perp \phi_{GK}, \quad \left\langle \nabla_\parallel \langle \phi_{GK} \rangle_{\boldsymbol{R}} \right\rangle_{SC} = 0,$$

Only a small-scale perpendicular electric field affects the large-scale dynamics
Removes gyroaverage to satisfy the DK quasi neutrality $\nabla \cdot \boldsymbol{J} = 0$
Separate Boltzmann equation:

$$\langle \partial_t F_a \rangle_{SC} = \partial_t F_{aDK}, \quad \partial_t F_a - \langle \partial_t F_a \rangle_{SC} = \partial_t F_{aGK}$$

$$\partial_t \Omega = \nabla_\parallel \left(J_\parallel + \frac{1}{2 m_i \Omega_i^2} P_{\perp iDK} \nabla_\perp^2 U_{\parallel i} \right) + \frac{1}{B\Omega_i} \nabla_\perp S_N \cdot \nabla_\perp \phi_{DK}$$
$$- \nabla_\perp \cdot \left(\nabla_\parallel \left(N_{iDK} \boldsymbol{\omega} U_{\parallel i} \right) + \frac{1}{B} \left[\phi_{DK} + \phi_{GK}, N_{iDK} \boldsymbol{\omega} \right] \right)$$
$$+ \frac{1}{2 m_i \Omega_i^2} \left[\nabla_\perp^2 \phi_{GK}, P_{\perp iDK} \right] - \frac{1}{\Omega_i B^2} \left[N_{iDK} \nabla_\perp \phi_{GK}, \nabla_\perp \phi_{DK} \right].$$
$$\boldsymbol{\omega} = \frac{1}{B\Omega_i} \nabla_\perp \phi_{DK} + \frac{1}{m_i \Omega_i^2 N_{iDK}} \nabla_\perp P_{\perp iDK}, \quad \Omega = \nabla \cdot \left(N_i \boldsymbol{\omega} \right),$$
$$J_\parallel = N_{eDK} \left(U_{\parallel i1} - U_{\parallel e1} \right),$$

Constraints

$$N_{iDK} = N_{eDK} + \mathcal{O}\left(N_{eDK} \epsilon_\perp^2 \right), \quad \nabla_\parallel U_{\parallel i} = \nabla_\parallel U_{\parallel e} + \mathcal{O}\left(\nabla_\parallel U_{\parallel e} \epsilon_\perp^2 \right),$$

$$N_{iDK} = N_{eDK} + \mathcal{O}\left(N_{eDK}\epsilon_\perp^2\right), \quad \nabla_\parallel U_{\parallel i} = \nabla_\parallel U_{\parallel e} + \mathcal{O}\left(\nabla_\parallel U_{\parallel e}\epsilon_\perp^2\right),$$

Impose

$$U_{\parallel i} = U_{\parallel e},$$

Total momentum equation (lowest $\epsilon_\perp^2$ order)

$$m_i N_e \partial_t U_{\parallel i} + m_e N_e \partial_t U_{\parallel e} = \dots$$

$J_\parallel$ is evolved as dynamical field, imposes (truncation)

$$\frac{e}{m_e} N_{eDK} \nabla_\parallel \phi_{DK} - \frac{1}{m_e}\nabla_\parallel P_{\parallel eDK} + \frac{1}{m_i}\nabla_\parallel P_{\parallel iDK} = \mathcal{O}\left(N_{eDK}\partial_t U_{\parallel e}\epsilon_\perp^2\right)$$

Parallel acceleration balance for retaining quasi neutrality

DK electron Drift-reduced Braginskii (critical balance)

$$\partial_t \mathcal{N}^{pj}_{iDK} + \nabla_\parallel \left\|v_\parallel\right\|^{pj}_{iDK} + \sqrt{2p}\left\|s_{\parallel i}\right\|^{p-1j}_{iDK} \nabla_\parallel U_\parallel + \frac{1}{B}\left[\phi_{DK} + \phi_{GK}, \mathcal{N}^{pj}_{iDK}\right]$$
$$- \frac{\sqrt{2p}\mathcal{N}^{p-1j}_{iDK}}{N_{eDK}} \frac{1}{m_i v_{Thi}} \nabla_\parallel P_{\parallel iDK} = C^{pj}_{iDK} + S^{pj}_{iDK},$$

ϕ_{DK} evaluated through Ω evolved with $J_\parallel$ as dynamical fields
For $\nu_{ii} \gg \omega$ and $\phi_{GK} = 0$, Drift-reduced Braginskii is recovered
Blue: parallel advection, acceleration, Landau damping
Red: $\boldsymbol{E} \times \boldsymbol{B}$-like terms

Orange: collision and sources

$$\frac{\partial}{\partial t}\mathcal{N}^{pj}_{iGK} + \sqrt{\frac{2p}{v_{Thi}}}\left\|\dot{\boldsymbol{R}}\right\|^{p-1j}_{iDK1}\cdot\nabla U_{\parallel} + \nabla\cdot\left\|\dot{\boldsymbol{R}}\right\|^{pj}_{iDK1} - \sqrt{\frac{2p}{v_{Thi}}}\left\|\dot{v}_{\parallel}\right\|^{p-1j}_{iDK1}$$

$$+\sqrt{\frac{2p}{v_{Thi}}}\mathcal{N}^{p-1j}_{iGK}\frac{\partial}{\partial t}U_{\parallel} + \sqrt{\frac{2p}{v_{Thi}}}\left\|\dot{\boldsymbol{R}}\right\|^{p-1j}_{iGK}\cdot\nabla U_{\parallel} + \nabla\cdot\left\|\dot{\boldsymbol{R}}\right\|^{pj}_{iGK} - \sqrt{\frac{2p}{v_{Thi}}}\left\|\dot{v}_{\parallel}\right\|^{p-1j}_{iGK}$$

$$= C^{pj}_{iGK} + S^{pj}_{iGK},$$

$$\left\|\dot{\boldsymbol{R}}\right\|^{pj}_{iDK1}\cdot\nabla U_{\parallel} = \frac{1}{B}\left\|\boldsymbol{b}\times\nabla\left(\langle\psi\rangle_{\boldsymbol{R}} - \langle\phi_{GK}\rangle_{\boldsymbol{R}}\right)\right\|^{pj}_{iDK}\cdot\nabla U_{\parallel} + \frac{1}{B}\left\|\boldsymbol{b}\times\nabla\left\langle\widetilde{\phi_{GK}}\right\rangle_{\boldsymbol{R}}\right\|^{pj}_{iDK}\cdot\nabla U_{\parallel},$$

$$\nabla\cdot\left\|\dot{\boldsymbol{R}}\right\|^{pj}_{iDK1} = \frac{1}{B}\nabla\cdot\left\|\boldsymbol{b}\times\nabla\left(\langle\psi\rangle_{\boldsymbol{R}} - \langle\phi_{GK}\rangle_{\boldsymbol{R}}\right)\right\|^{pj}_{iDK} + \frac{1}{B}\nabla\cdot\left\|\boldsymbol{b}\times\nabla\left\langle\widetilde{\phi_{GK}}\right\rangle_{\boldsymbol{R}}\right\|^{pj}_{iDK},$$

$$\left\|\dot{v}_{\parallel}\right\|^{pj}_{iDK1} + \left\|\dot{v}_{\parallel}\right\|^{pj}_{iGK} = -\frac{q_i}{m_i}\left(\left\|\nabla_{\parallel}\langle\phi_{GK}\rangle_{\boldsymbol{R}}\right\|^{pj}_{iDK} + \mathcal{N}^{pj}_{iGK}\nabla_{\parallel}\phi_{DK}\right),$$

$$\nabla\cdot\left\|\dot{\boldsymbol{R}}\right\|^{pj}_{iGK} = \nabla_{\parallel}\left\|v_{\parallel}\right\|^{pj}_{iGK} + \frac{1}{B}\left[\phi_{DK},\mathcal{N}^{pj}_{iGK}\right] + \frac{1}{B}\nabla\cdot\left\|\boldsymbol{b}\times\nabla\langle\phi_{GK}\rangle_{\boldsymbol{R}}\right\|^{pj}_{iGK},$$

$$\langle\psi\rangle_{\boldsymbol{R}} = \langle\phi_{GK}\rangle_{\boldsymbol{R}} + \frac{q_a^2}{2m_a\Omega_a}\partial_{\mu}\left(\langle\phi_{GK}\rangle^2_{\boldsymbol{R}} - \left\langle\phi^2_{GK}\right\rangle_{\boldsymbol{R}}\right) + \frac{q_a}{2m_a\Omega_a^2}\left\langle\left(\mathrm{b}\times\nabla\widetilde{\phi_{GK}}\right)\cdot\nabla\widetilde{\phi_{GK}}\right\rangle_{\boldsymbol{R}}$$

Adiabatic GK electron response

$$
\begin{aligned}
F_{eGK} &= \frac{e\phi_{GK}}{T_{eDK}} F_{eDK}, \\
\varrho^*_{iGK} - \frac{e\phi_{GK}}{T_{eDK}} N_{eDK} &= -\frac{q_i}{B} \left\| \partial_\mu \left\langle \left\langle \phi_{GK} \right\rangle_{\boldsymbol{R}} \right\rangle^\dagger_{\boldsymbol{x}} \right\|_{iDK} \\
\varrho^*_{iGK} &= \left\| \left\langle F_{iGK} \right\rangle_{\boldsymbol{R}} / F_{iGK} \right\|_{iGK}
\end{aligned}
$$

For $F_{aDK}(\boldsymbol{R}, \boldsymbol{v}, t) = F_{Ma}(\boldsymbol{R}, \boldsymbol{v})$, and $\phi_{DK} = 0$, δ-f GK is recovered

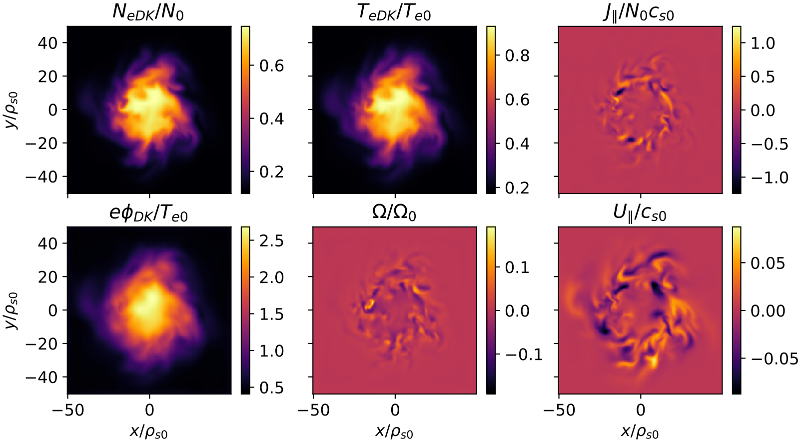
DK fields

- At max 2. order derivatives
- Finite differences for $\nabla_{\parallel}$ works
- Invert $\nabla_{\perp} \cdot (N_{eDK} \nabla_{\perp})$ Poisson operator
- Arakawa scheme for $[\cdot, \cdot]$
- Finite differences in perpendicular plane

GK fields

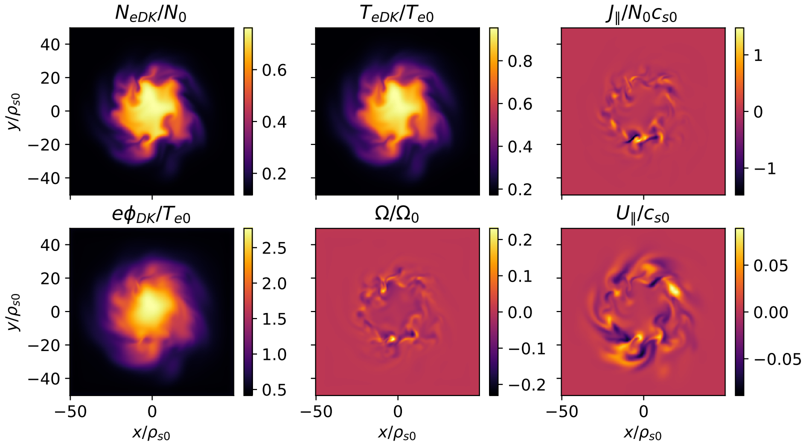
- Higher order $\perp$ derivatives
- Finite differences for $\nabla_{\parallel}$ works
- Invert $\sum_j \mathcal{N}_{iDK}^{0j} k_{\perp GK}^{2(j+1)}$ Poisson operator
- $\langle \cdot \rangle_{\boldsymbol{R}} = J_0 \left(v_{\perp} k_{\perp} / \Omega_i \right) \cdot$ in Fourier
- Fourier in perpendicular plane

With GK fields on

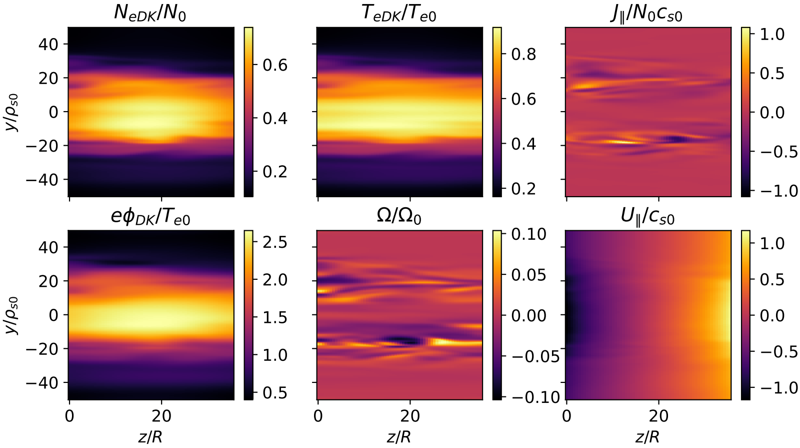


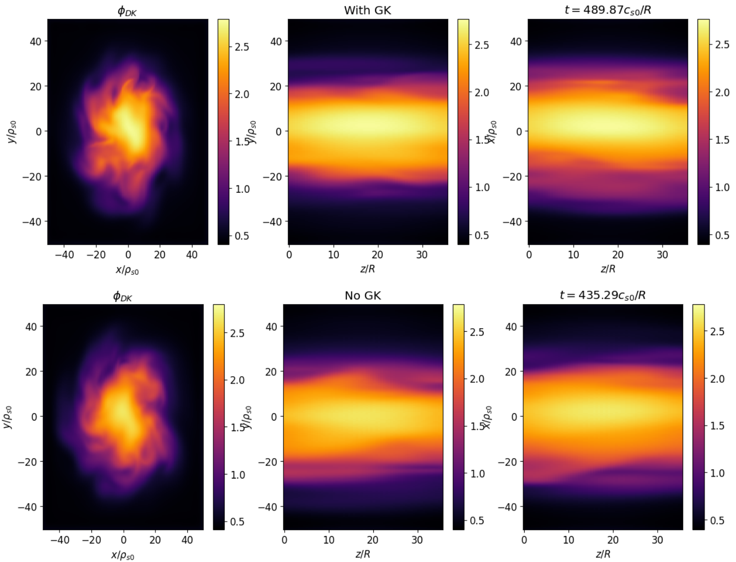
Swiss Plasma Center

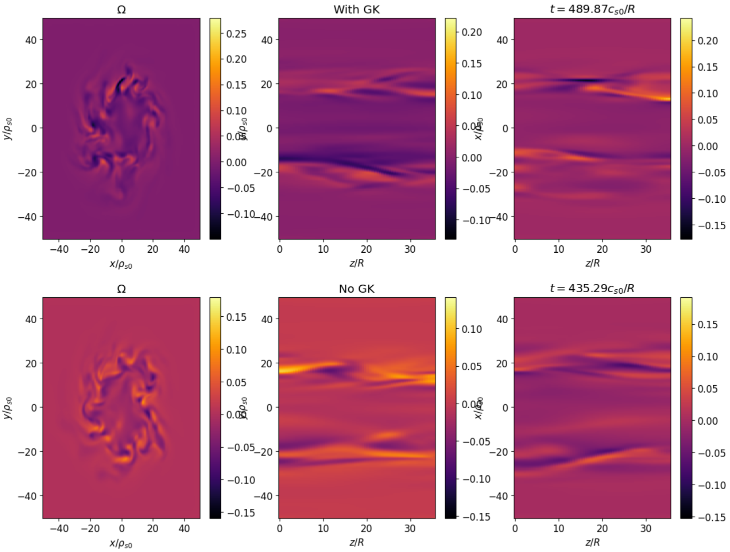
Without GK fields



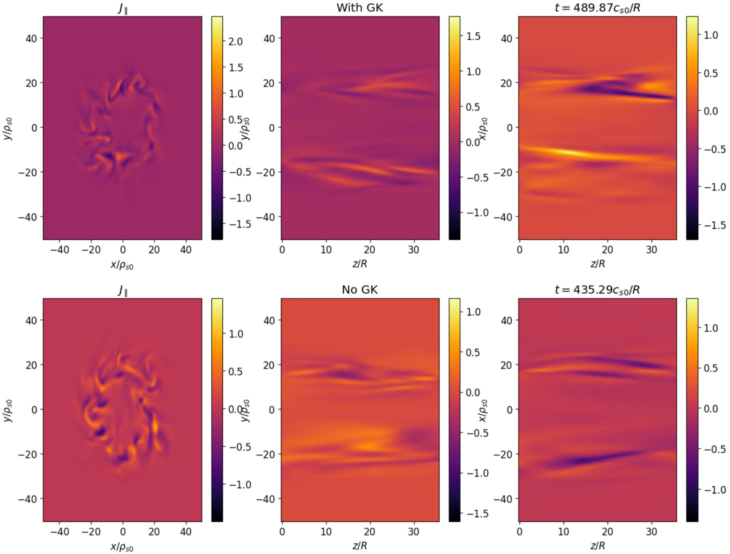
With GK fields on



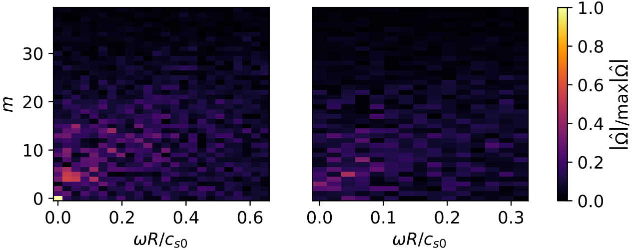

ϕ_{DK}
With GK
$t = 489.87 c_{s0}/R$
ϕ_{DK}
No GK
$t = 435.29 c_{s0}/R$
x/ρ_{s0}
y/ρ_{s0}
z/R
x/ρ_{s0}



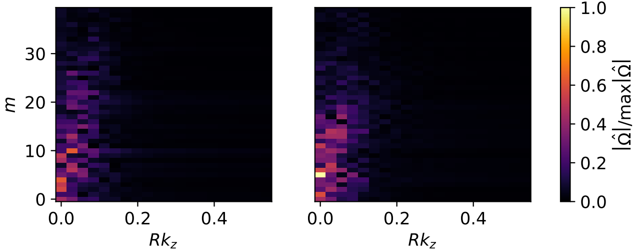
Ω
With GK
t = 489.87c_s0/R
Ω
No GK
t = 435.29c_s0/R
x/ρ_s0
y/ρ_s0
z/R
x/ρ_s0

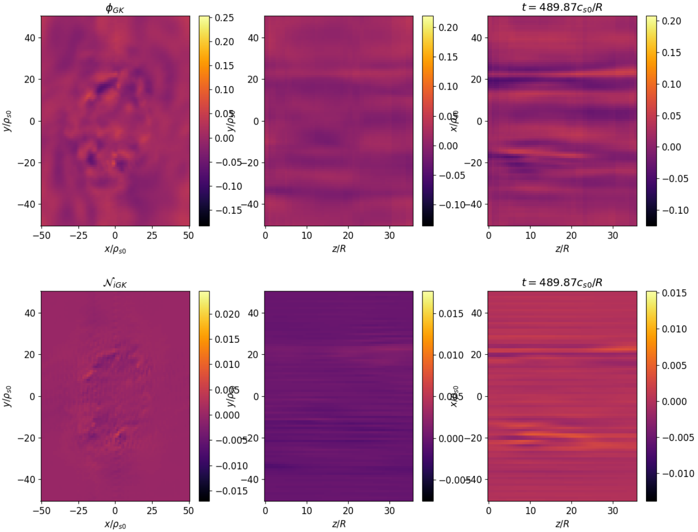

$J_\parallel$
With GK
$t = 489.87 c_{s0}/R$
$J_\parallel$
No GK
$t = 435.29 c_{s0}/R$
x/ρ_{s0}
y/ρ_{s0}
z/R
x/ρ_{s0}

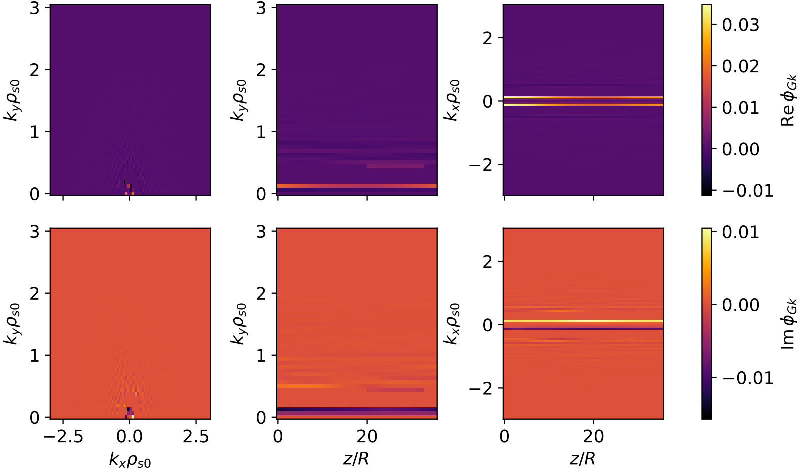
$$\Omega = \sum_{m,\omega} \hat{\Omega} e^{i(m\vartheta - \omega t)} \text{ at } r = \sqrt{x^2 + y^2} = 20\rho_{s0}$$

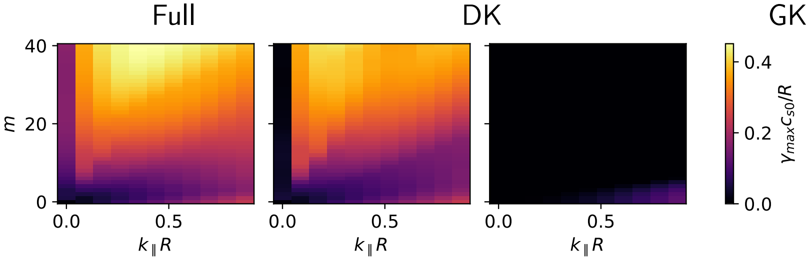


$$\Omega = \sum_{m,\omega} \hat{\Omega} e^{i(m\vartheta + k_z z)} \text{ at } r = \sqrt{x^2 + y^2} = 20\rho_{s0}$$



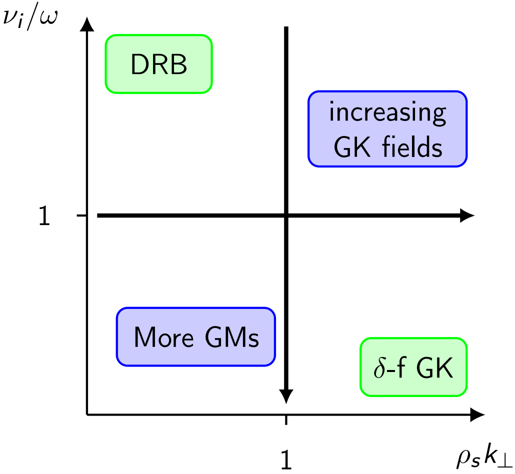

ϕ_{GK}
$t = 489.87 c_{s0}/R$
$\mathcal{N}_{iGK}$
$t = 489.87 c_{s0}/R$
y/ρ_{s0}
x/ρ_{s0}
z/R


3
2
1
0
$k_y\rho_{s0}$
$k_x\rho_{s0}$
−2.5
0.0
2.5
z/R
0
20
−2
$\mathrm{Re}\,\phi_{Gk}$
0.03
0.02
0.01
0.00
−0.01
$\mathrm{Im}\,\phi_{Gk}$



- DK: KH modes dominate
- Full: amplified KH
- GK: $\|\partial_z \langle \phi_{GK} \rangle_{\boldsymbol{R}}\|^{pj}$ modes

- GK fields cause slight increase in high $\rho_s k_\perp$ modes
- High ν_{ii} diminishes GK effect $C_{iiGK} \sim \mu k_\perp^2 F_{iGK}$
- Go to tokamak geometry (large apect ratio, $B_\phi \gg B_\theta$)



$$S_i = \langle A_N(\boldsymbol{x})\rangle_{\boldsymbol{R}} F_{Mi} + \langle A_E(\boldsymbol{x})\rangle_{\boldsymbol{R}} \left(s_{\parallel i}^2 + x_i - \frac{3}{2}\right) N_{eDK} F_{Mi},$$

$$S_{Te} = A_{Te}(\boldsymbol{R}),$$

$$\begin{aligned}\mathcal{C}_{ii0}^{pj} =& \nu_i \Big[-(p+2j)\mathcal{N}_{iDK}^{pj} + (T_{iDK}-1) \\ & \times \left(\sqrt{p(p-1)}\mathcal{N}_{iDK}^{p-2j} - 2j\mathcal{N}_{iDK}^{pj-1}\right)\Big] \\ \mathcal{C}_{ii1}^{pj} =& \nu_{i0} \Big[-(p+2j)\mathcal{N}_{iGK}^{pj} + \tau_i T_{iDK}\nabla_\perp^2 \mathcal{N}_{iGK}^{pj} + (T_{iDK}-1) \\ & \times \left(\sqrt{p(p-1)}\mathcal{N}_{iGK}^{p-2j} - 2j\mathcal{N}_{iGK}^{pj-1}\right)\Big].\end{aligned}$$

Convergence