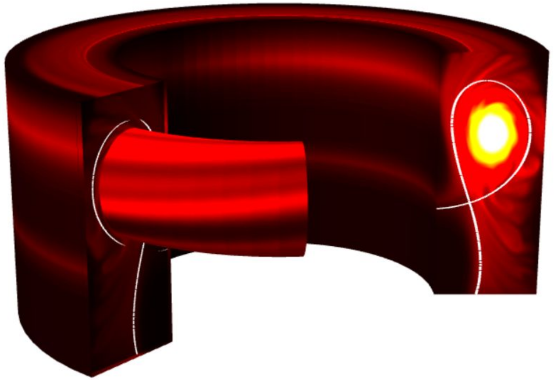


KINetic Deterministic NEutral Solver (KINDNES) - A kinetic neutrals solver that avoids Monte Carlo noise for turbulent simulations in magnetic confinement devices

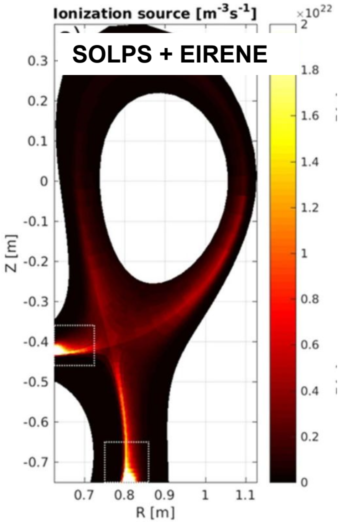
D. Mancini and P. Ricci

EPFL
SWISS PLASMA CENTER

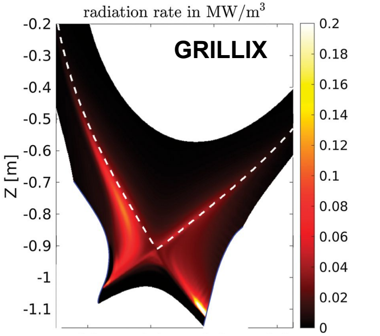
Plasma simulations need a self-consistent fast neutral model

Accurate description of neutral dynamics at the boundary of magnetic confinement devices is fundamental to understand:

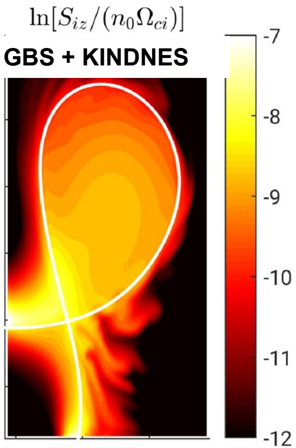
- Plasma recycling at the wall and heat exhaust
- Dissipative regimes like detachment *[D. Mancini et al, NF 2024]* or XPR *[K. Eder et al, ArXiv 2025]*
- Interplay between neutrals and **plasma turbulence** *[C. Wersal et al, NF 2017]*



[Y. Wang et al, NF 2024]



[W. Zholobenko et al, NF 2021]



[M. Giacomin et al, JCP 2021]

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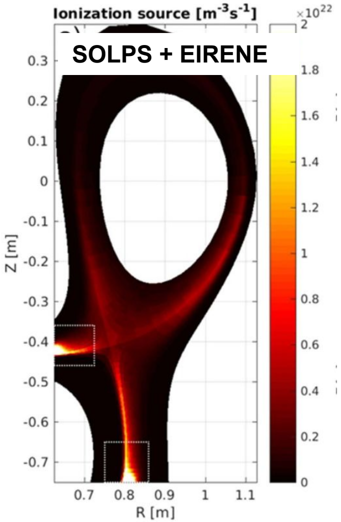

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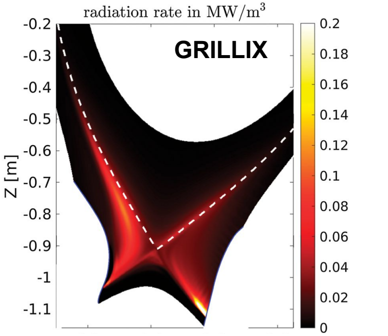
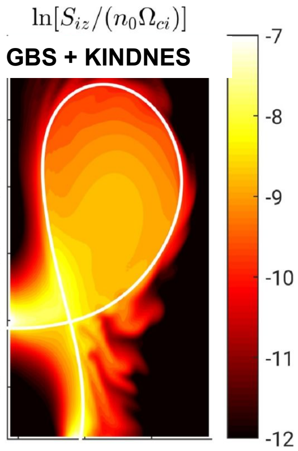


[W. Zholobenko et al, NF 2021]



[M. Giacomin et al, JCP 2021]

Neutral codes are typically divided between:

- Fluid models (Hermes, nHesel, SolEdge2D, Grillix)
- Kinetic Monte Carlo models (Eirene, Degas2)

KINDNES allows a **kinetic** description **without Monte Carlo** methods

Index

- **The basic elements of the model**:
  - Method of characteristics for kinetic neutrals
  - Extension to multiple species
- On-going developments:
  - Flexible wall geometry
  - Memory reduction with hierarchical matrices
  - Full 3D reconstruction
  - Decoupling KINDNES from GBS

Boltzmann equation for neutral particles (one species)

Boltzmann equation for atomic Deuterium:

$$\frac{\partial f_D}{\partial t}+\mathbf{v}\cdot\frac{\partial f_D}{\partial \mathbf{x}} = -\nu_{iz} f_D - \nu_{cx}\left(f_D - \frac{n_D}{n_{D^+}} f_{D^+}\right) + \nu_{rec} f_{D^+}$$

- Different reactions leads to creation/removal of neutrals
- PDE with:
  - decay rates proportional to f_D
  - a source term independent of f_D

[C. Wersal and P. Ricci 2015 NF]

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$$\mathrm{e}^- + \mathrm{D} \rightarrow 2\mathrm{e}^- + \mathrm{D}^+$$

Ionization

$$\mathrm{D}^+ + \mathrm{D} \rightarrow \mathrm{D} + \mathrm{D}^+$$

Charge-Exchange

$$\mathrm{e}^- + \mathrm{D}^+ \rightarrow \mathrm{D}$$

Recombination

- Different reactions leads to creation/removal of neutrals
- PDE with:
  - decay rates proportional to f_D
  - a source term independent of f_D (but dependent on n_D)

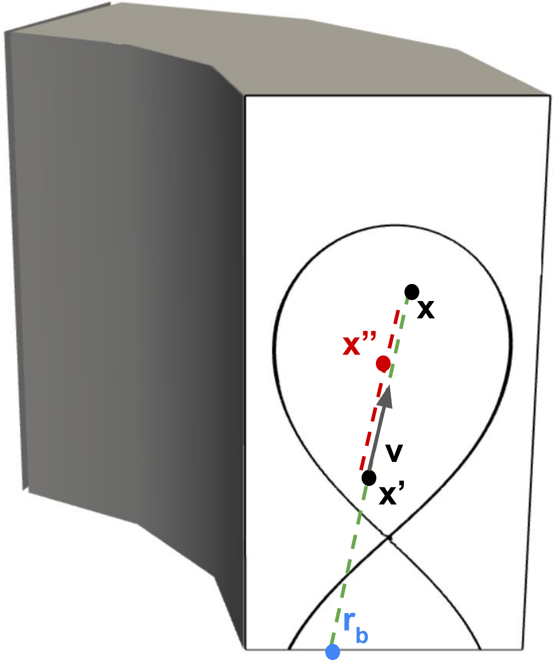
[C. Wersal and P. Ricci 2015 NF]

Solution based on deterministic algorithm avoiding statistical noise of Monte Carlo methods

$$\frac{\partial f_D}{\partial t}+\mathbf{v}\cdot\frac{\partial f_D}{\partial \mathbf{x}} = -\nu_{iz} f_D - \nu_{cx}\left(f_D - \frac{n_D}{n_{D^+}} f_{D^+}\right) + \nu_{rec} f_{D^+}$$

Formal solution with method of characteristics (neutrals move on straight paths):

$$f_D(\mathbf{x},\mathbf{v},t) = \int_0^{r_b}\left[\frac{S_D(\mathbf{x}',\mathbf{v},t)}{v} + \delta(r'-r_b')f_D(\mathbf{x}_\mathbf{b}',\mathbf{v},t_b')\right]\exp\left(-\frac{1}{v}\int_0^{r'}\nu_{\mathrm{eff},D}(\mathbf{x}'',t'')dr''\right)dr'$$



[C. Wersal and P. Ricci 2015 NF]

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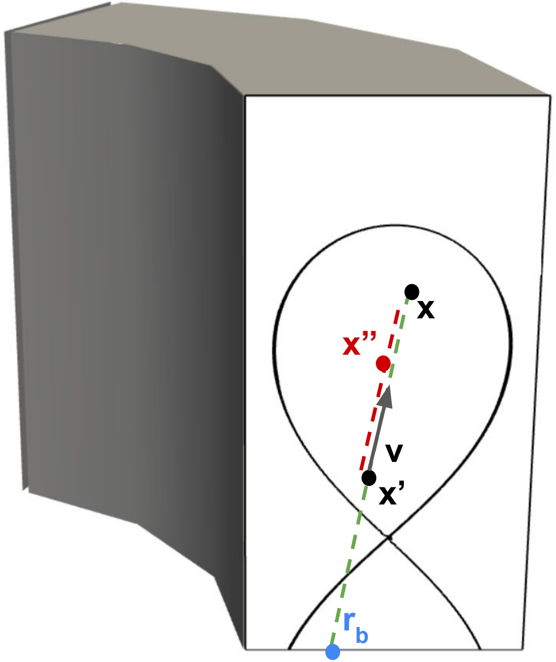


- Volumetric source $S_D = \nu_{cx} n_D \frac{f_{D^+}}{n_{D^+}} + \nu_{rec} f_{D^+}$
- Boundary condition
- Decay rate $\nu_{\text{eff},D} = \nu_{iz} + \nu_{cx}$

[C. Wersal and P. Ricci 2015 NF]

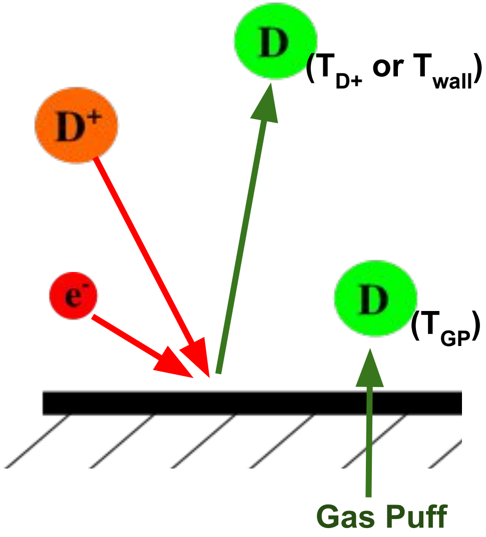
Boundary conditions describe emission, reflection and puffing

For the points on the boundary $\boldsymbol{x}_b$:

$$f_D(\mathbf{x}_b,\mathbf{v}) = (1-\alpha_{\rm refl})\Gamma_{\rm out}\chi_{\rm in}(\mathbf{x}_b,\mathbf{v}) + \alpha_{\rm refl}[f_D(\mathbf{x}_b,\mathbf{v}-2\mathbf{v}_\perp)+f_{D^+}(\mathbf{x}_b,\mathbf{v}-2\mathbf{v}_\perp)] + \Gamma_{{\rm GP},D}\chi_{GP}(\mathbf{x}_b,\mathbf{v})$$

Boundary condition at the wall describes:

- Neutral recycling due to **ion** and **neutral** flux to the wall $\Gamma_{\rm out} = \Gamma_{{\rm out},D} + \Gamma_{{\rm out},D^+}$
- Reflection (ion temperature) and re-emission (χ_{in})
- Additional sources like gas puff (χ_{GP})



[C. Wersal and P. Ricci 2015 NF]

Two approximations to simplify the solution

$$f_D(\mathbf{x},\mathbf{v},t) = \int_0^{r_b} \left[\frac{S_D(\mathbf{x}',\mathbf{v},t)}{v} + \delta(r'-r'_b) f_D(\mathbf{x}'_\mathbf{b},\mathbf{v},t'_b)\right] \exp\left(-\frac{1}{v}\int_0^{r'} \nu_{\mathrm{eff},D}(\mathbf{x}'',t'')dr''\right) dr'$$

- Neutral adiabatic regime $1/\nu_{\mathrm{eff}} < \tau_{\mathrm{turb}}$ $\longrightarrow$ $\partial_t f_D = 0$
- Turbulence anisotropy $\lambda_{\mathrm{mfp},D} \ll 1/k_{||}$ $\longrightarrow$ Single plane solution

Solution algorithm for neutral density

Integrating in velocity space for the first moment of the distribution:

$$n_{\mathrm{D}} = \int d\mathbf{v}\ f_D(\mathbf{x},\mathbf{v},t) = \int\int_0^{r_b} \left[\frac{S_D(\mathbf{x}',\mathbf{v},t)}{v} + \delta(r'-r'_b) f_D(\mathbf{x}'_{\mathbf{b}},\mathbf{v},t'_b)\right] \exp\left(-\frac{1}{v}\int_0^{r'} \nu_{\mathrm{eff},D}(\mathbf{x}'',t'')dr''\right) dr'\ d\mathbf{v}$$

Using the explicit boundary condition:

$$n_D(\mathbf{x}) = \int_\Sigma n_D(\mathbf{x}')\nu_{\mathrm{cx}}(\mathbf{x}')K_{p\to p}(\mathbf{x},\mathbf{x}')d\Sigma + \int_{\partial\Sigma} \Gamma_{\mathrm{out},D}(\mathbf{x}'_b)K_{b\to p}(\mathbf{x},\mathbf{x}'_b)d\sigma_b + n_{D,\mathrm{rec}}(\mathbf{x}) + n_{D,\mathrm{GP}}(\mathbf{x})$$

$$\Gamma_{\mathrm{out},D}(\mathbf{x}_b) = \int_\Sigma n_D(\mathbf{x}')\nu_{\mathrm{cx}}(\mathbf{x}')K_{p\to b}(\mathbf{x}_b,\mathbf{x}')d\Sigma + \int_{\partial\Sigma} \Gamma_{\mathrm{out},D}(\mathbf{x}'_b)K_{b\to b}(\mathbf{x}_b,\mathbf{x}'_b)d\sigma_b + \Gamma_{\mathrm{out,rec}}(\mathbf{x}_b) + \Gamma_{\mathrm{out,D^+}}(\mathbf{x}_b) + \Gamma_{\mathrm{out,GP}}(\mathbf{x}_b)$$

Solution algorithm for neutral density

EPFL

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- System of two integral equations to evaluate n_D
- Velocity integrals hidden in the 4 kernel functions K_{pp}, K_{pb}, K_{bp}, K_{bb}

Model solved plane by plane on discrete grid

$$n_D(\mathbf{x}) = \int_\Sigma n_D(\mathbf{x}')\nu_{\text{cx}}(\mathbf{x}')K_{p\to p}(\mathbf{x},\mathbf{x}')d\Sigma + \int_{\partial\Sigma} \Gamma_{\text{out},D}(\mathbf{x}_b')K_{b\to p}(\mathbf{x},\mathbf{x}_b')d\sigma_b + n_{D,\text{rec}}(\mathbf{x}) + n_{D,\text{GP}}(\mathbf{x})$$

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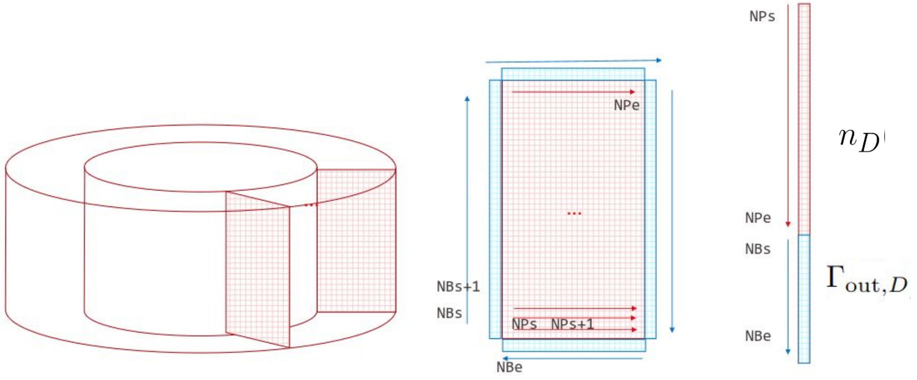
Discretized as a system on a linearized grid:

$$\begin{bmatrix} n_D \\ \Gamma_{\text{out},D} \end{bmatrix} = \begin{bmatrix} \nu_{cx}K_{pp} & K_{bp} \\ \nu_{cx}K_{pb} & K_{bb} \end{bmatrix} \begin{bmatrix} n_D \\ \Gamma_{\text{out},D} \end{bmatrix} + \begin{bmatrix} n_{D,\text{rec}} + n_{D,\text{GP}} \\ \Gamma_{\text{out,rec}} + \Gamma_{\text{out},D^+} + \Gamma_{\text{out,GP}} \end{bmatrix}$$



Model solved plane by plane on discrete grid

Discretized as a system on a linearized grid:

$$\underbrace{\begin{bmatrix} n_D \\ \Gamma_{\text{out},D} \end{bmatrix}}_{\text{x}} = \underbrace{\begin{bmatrix} \nu_{cx} K_{pp} & K_{bp} \\ \nu_{cx} K_{pb} & K_{bb} \end{bmatrix}}_{K} \begin{bmatrix} n_D \\ \Gamma_{\text{out},D} \end{bmatrix} + \underbrace{\begin{bmatrix} n_{D,\text{rec}} + n_{D,\text{GP}} \\ \Gamma_{\text{out,rec}} + \Gamma_{\text{out},D^+} + \Gamma_{\text{out,GP}} \end{bmatrix}}_{\text{b}}$$

- Neutral density from linear system inversion for each plane: $(\mathbb{1} - K)\,\text{x} = \text{b}$
- Integrating the previous equation for other moments yields $\boldsymbol{\Gamma}_{D,\perp}, \boldsymbol{\Gamma}_{D,\parallel}, T_D$

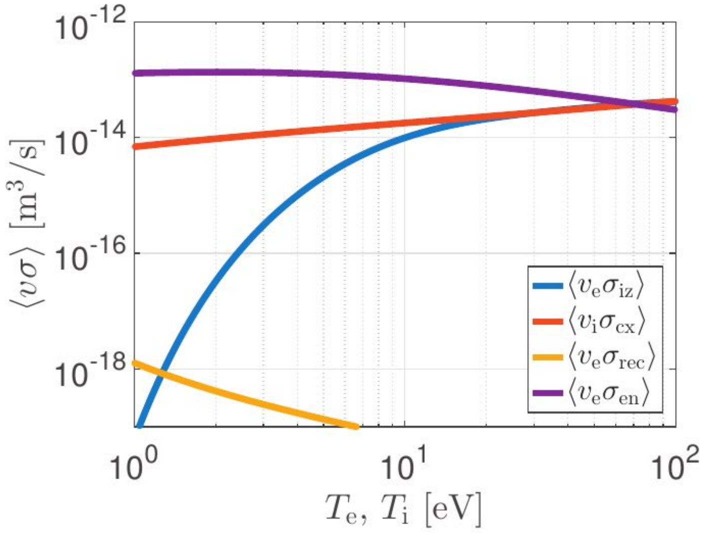
Implementation:

- Easy iterative solver with Petsc libraries
- Highly parallelizable → plane by plane + each plane parallelized by Petsc

KINDNES was born coupled with GBS

The current GBS code structure is:

- Short cycling scheme: neutrals solved every *N* plasma steps (typically *100 < N < 5000*)
  - Plasma density, temperature and fluxes to the wall interpolated on a cartesian neutral grid
  - System solved on the neutral grid
  - Neutral moments interpolated on plasma grid

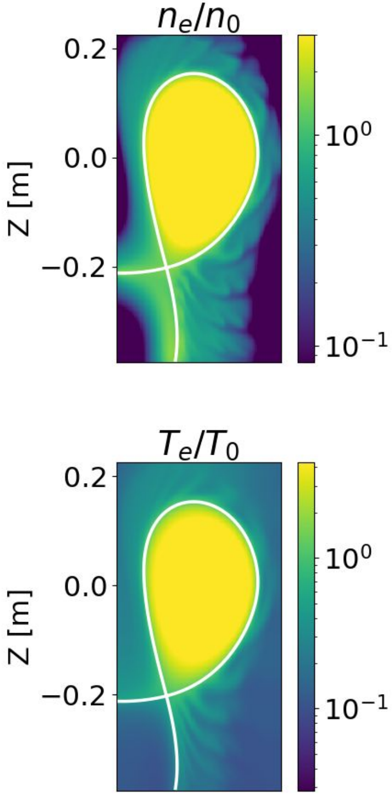


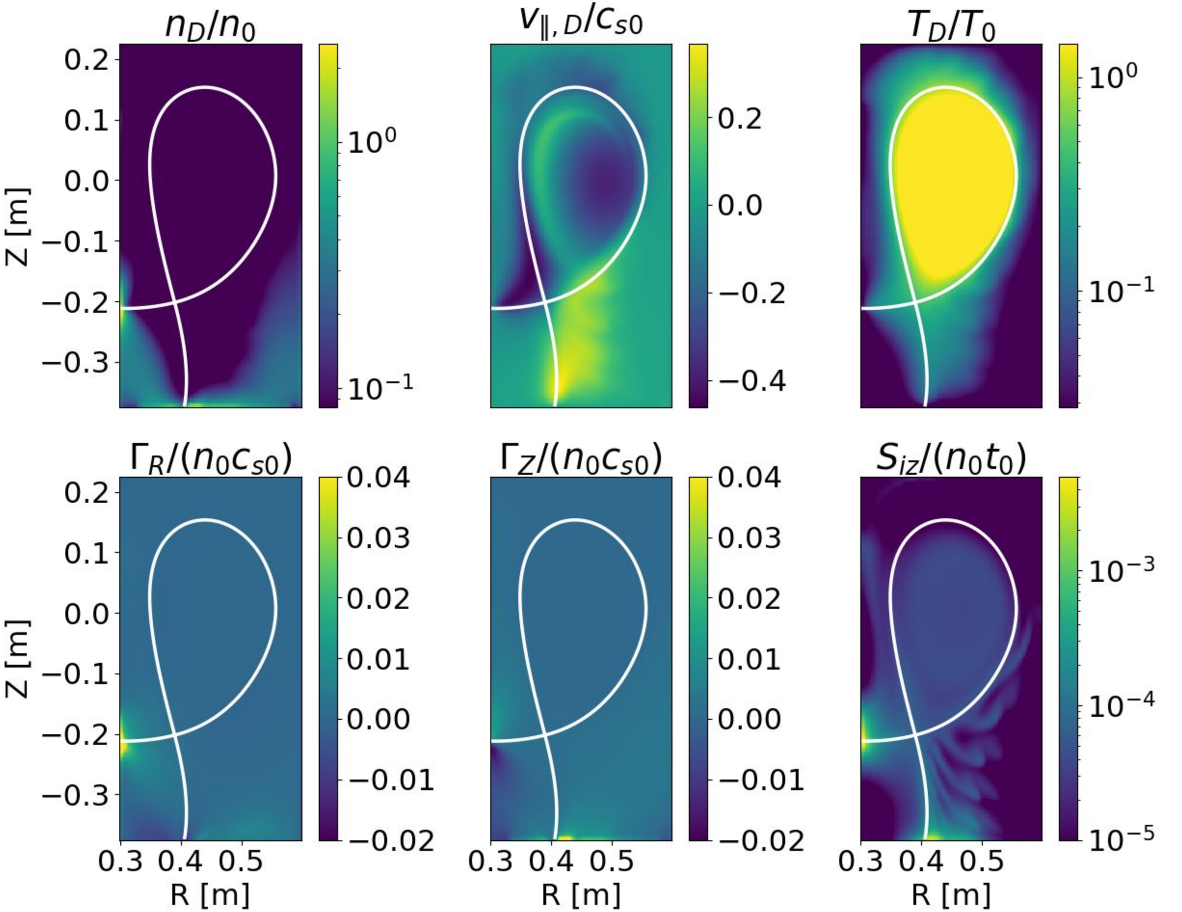
- Plasma sources updated at every step with reaction rates evaluated as Krook operators $\nu_{\mathrm{iz,D}} = n\langle \sigma_{\mathrm{iz,D}} v \rangle (n, T_e)$

[C. Wersal and P. Ricci 2015 NF , OpenAdas data http://open.adas.ac.uk]

Example of the solution in a standard TCV-X21 attached L-mode case

Background plasma:



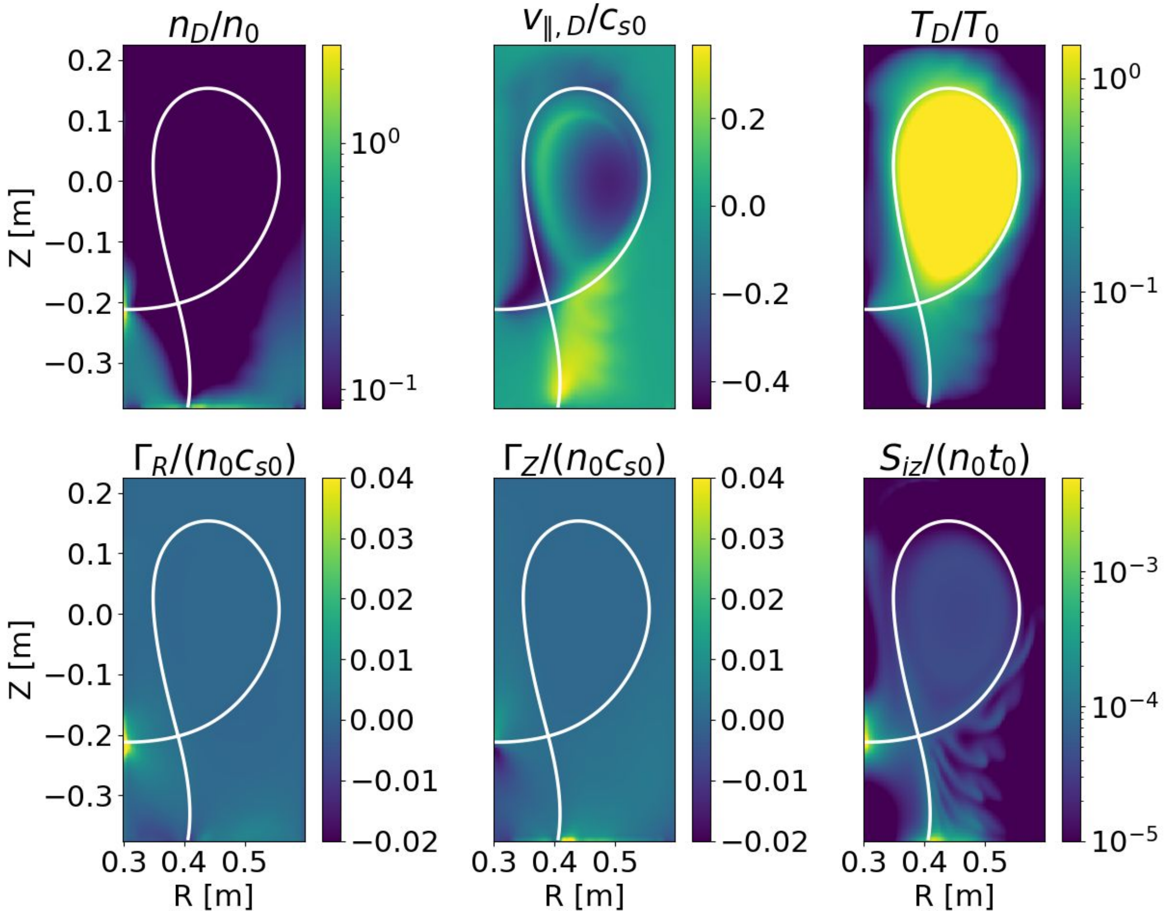


Example of the solution in a standard TCV-X21 attached L-mode case

Some numbers:

- Plasma grid 150 x 300 x 64
- Neutrals grid 50 x 100 x 64
- Time to solution on
  5 x 4 x 64 cpu (Pitagora)
  = 16s (one neutral update
  every 1000 plasma step)

Grid resolution scalings
and unit testing of the code
described in
[M. Giacomin et al., JCP, 2022]



Extension to multiple species: adding molecular Deuterium D_2

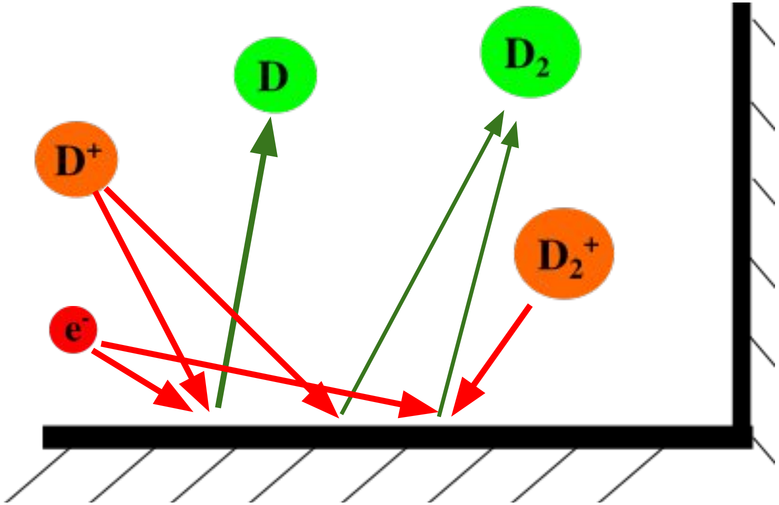
Two coupled PDEs:

$$\frac{\partial f_{D^+}}{\partial t} + \mathbf{v}\cdot\frac{\partial f_{D^+}}{\partial \mathbf{x}} = \nu_{iz,D} f_D - \nu_{rec,D^+} f_{D^+} - \nu_{cx,D}\left(\frac{n_D}{n_{D^+}} f_{D^+} - f_D\right) + \nu_{cx,D\text{-}D_2^+} f_D - \nu_{cx,D_2-D^+}\frac{n_{D_2}}{n_{D^+}} f_{D^+} + \ldots$$

$$\frac{\partial f_{D_2^+}}{\partial t} + \mathbf{v}\cdot\frac{\partial f_{D_2^+}}{\partial \mathbf{x}} = \nu_{iz,D_2} f_{D_2} - \nu_{rec,D_2^+} f_{D_2^+} - \nu_{cx,D_2}\left(\frac{n_{D_2}}{n_{D_2^+}} f_{D_2^+} - f_{D_2}\right) - \nu_{cx,D_2-D^+} f_{D_2} - \nu_{cx,D\text{-}D_2^+}\frac{n_D}{n_{D_2^+}} f_{D_2^+} + \ldots$$

Boundary conditions:

- Neutral recycling due to total ion flux to the wall (including **parallel** and **drift velocity**)
- Reflection, re-emission, and association



[A. Coroado and P. Ricci 2022 NF]

Extension to multiple species: adding molecular Deuterium D_2

EPFL

$$\frac{\partial f_{D^+}}{\partial t} + \mathbf{v}\cdot\frac{\partial f_{D^+}}{\partial \mathbf{x}} = \nu_{iz,D} f_D - \nu_{rec,D^+} f_{D^+} - \nu_{cx,D}\left(\frac{n_D}{n_{D^+}} f_{D^+} - f_D\right) + \nu_{cx,D\text{-}D_2^+} f_D - \nu_{cx,D_2-D^+}\frac{n_{D_2}}{n_{D^+}} f_{D^+} + \ldots$$

$$\frac{\partial f_{D_2^+}}{\partial t} + \mathbf{v}\cdot\frac{\partial f_{D_2^+}}{\partial \mathbf{x}} = \nu_{iz,D_2} f_{D_2} - \nu_{rec,D_2^+} f_{D_2^+} - \nu_{cx,D_2}\left(\frac{n_{D_2}}{n_{D_2^+}} f_{D_2^+} - f_{D_2}\right) - \nu_{cx,D_2-D^+} f_{D_2} - \nu_{cx,D\text{-}D_2^+}\frac{n_D}{n_{D_2^+}} f_{D_2^+} + \ldots$$

The neutral densities can be obtained by inverting a large linear system:

$$\begin{pmatrix} n_D \\ \Gamma_{out,D} \\ n_{D_2} \\ \Gamma_{out,D_2} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} & 0 \\ M_{21} & M_{22} & M_{23} & 0 \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} \begin{pmatrix} n_D \\ \Gamma_{out,D} \\ n_{D_2} \\ \Gamma_{out,D} \end{pmatrix} + \begin{pmatrix} n_D^{rec,D^+} + n_D^{diss,D_2^+} \\ \Gamma_D^{rec,D^+} + \Gamma_{out,D}^{D^+}\Gamma_D^{refl,D^+} \\ n_{D_2}^{rec,D^+} \\ \Gamma_{D_2}^{rec,D_2^+} + \Gamma_{D_2}^{refl,D_2^+}\Gamma_{D_2}^{refl,D^+} \end{pmatrix}$$

Interactions with plasma (cx, iz, diss)

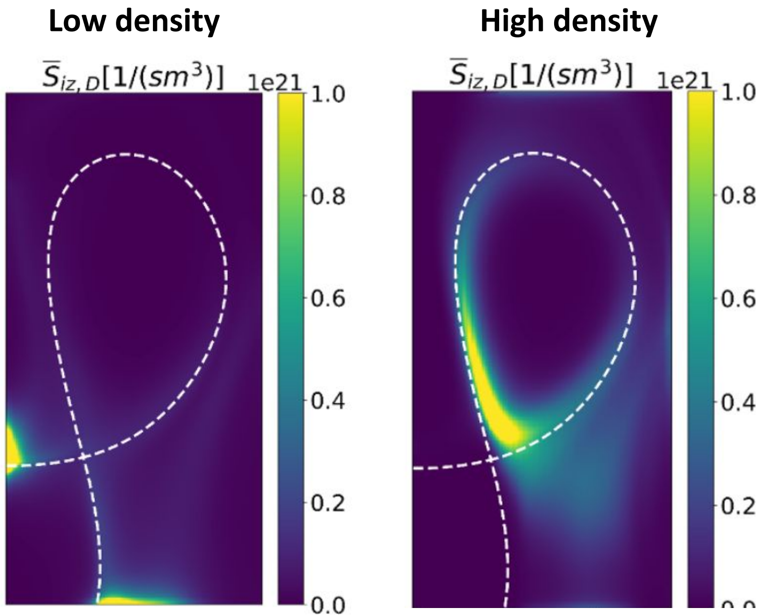
Neutrals from rec, diss and puffing

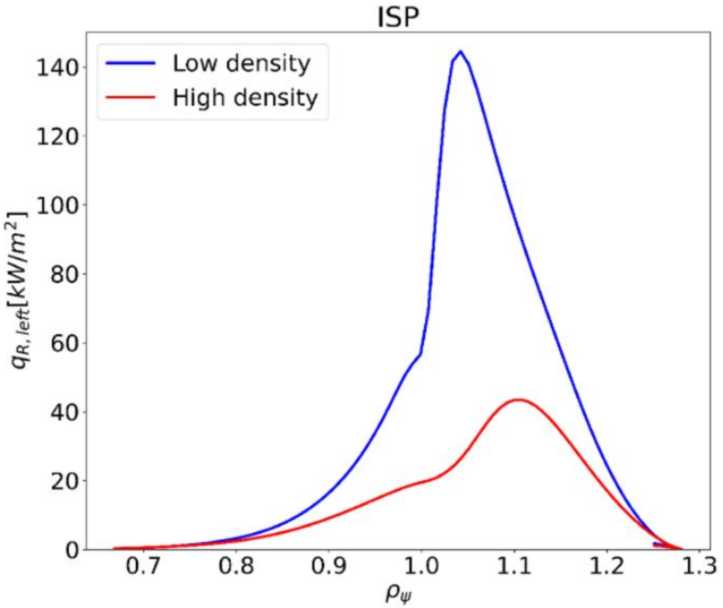
[A. Coroado and P. Ricci 2022 NF]

GBS simulations of detached plasma with molecules

Going from low to high density:

- Ionization front moves in the core
- Strong decrease of heat flux at the inner strike point (ISP) ⟶ detachment of inner target





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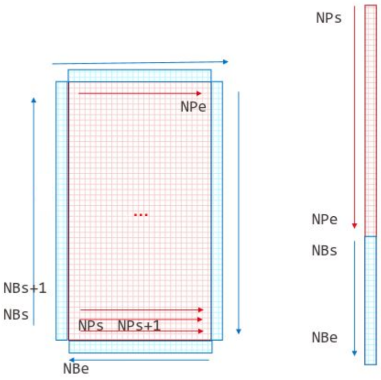
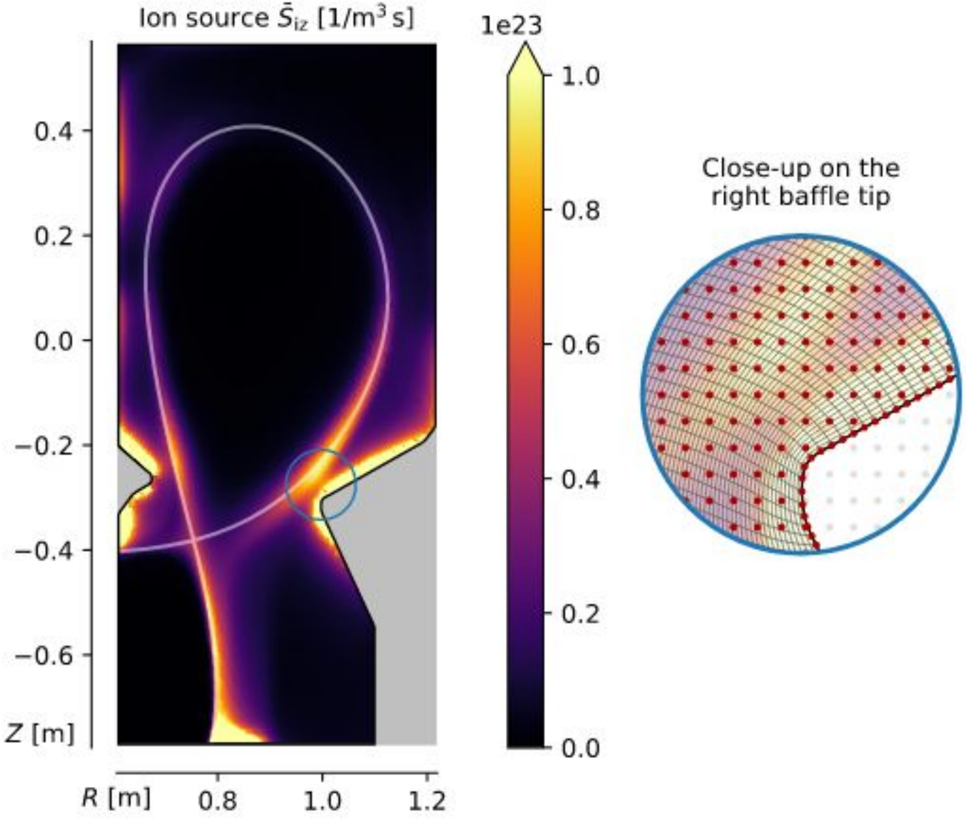
  - Memory reduction with hierarchical matrices
  - Full 3D reconstruction
  - Decoupling KINDNES from GBS

On-going: flexible wall geometry

Implementation on arbitrary boundary + cartesian grid:

- 🟢 Decoupling inner and boundary grid, given as inputs
- 🔴 Higher number of boundary points needed (slightly larger matrix to invert)



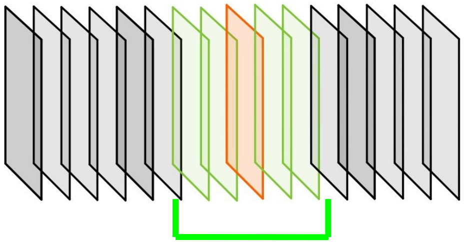


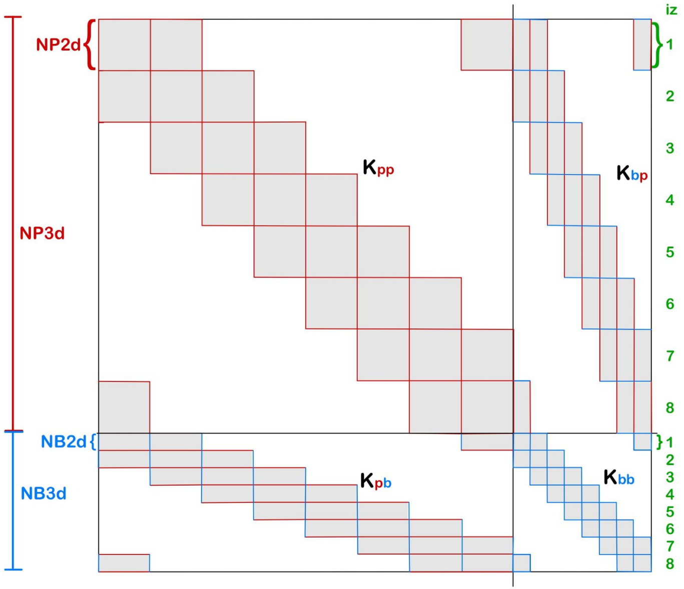
[L. Stenger et al, in preparation]

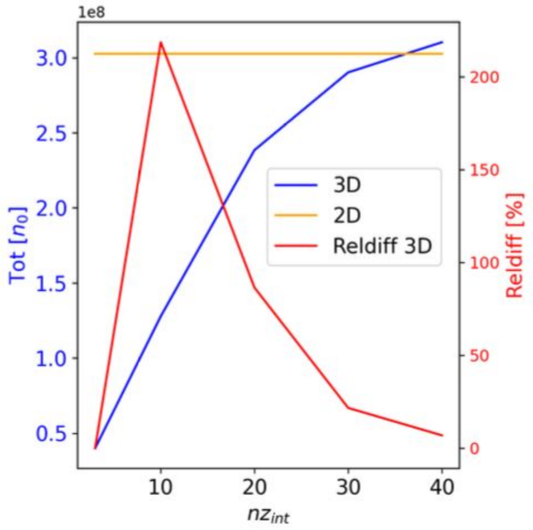
On-going: coupling between planes for non-uniform phenomena

Single solution for the whole tokamak:

- Drop of approximation $\lambda_{\mathrm{mfp},D} \ll 1/k_{\|}$
- Null elements for distant planes $\longrightarrow$ sparse matrix
- Large increase of matrix dimension







[G. Drago, EPFL Master thesis, 2025]

On-going: efficiency increased using hierarchical matrices (HM)

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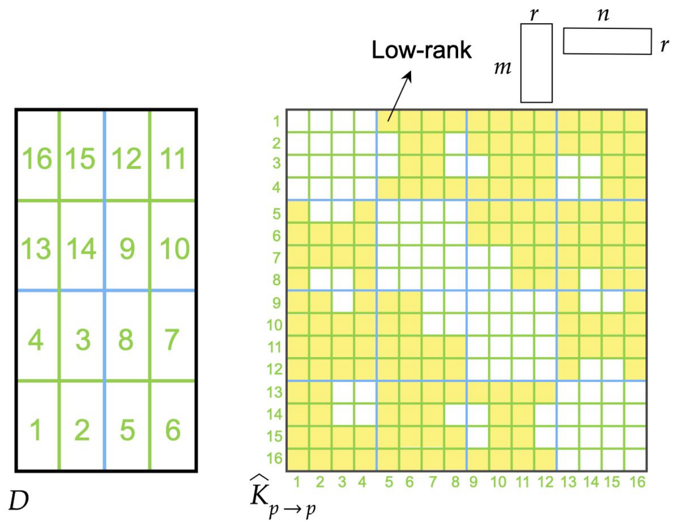
Approximation of the matrix elements when corresponding points are distant in space:

- Great reduction of number of elements evaluated $\longrightarrow$ reduced memory and time
- Allows high neutrals grid resolution

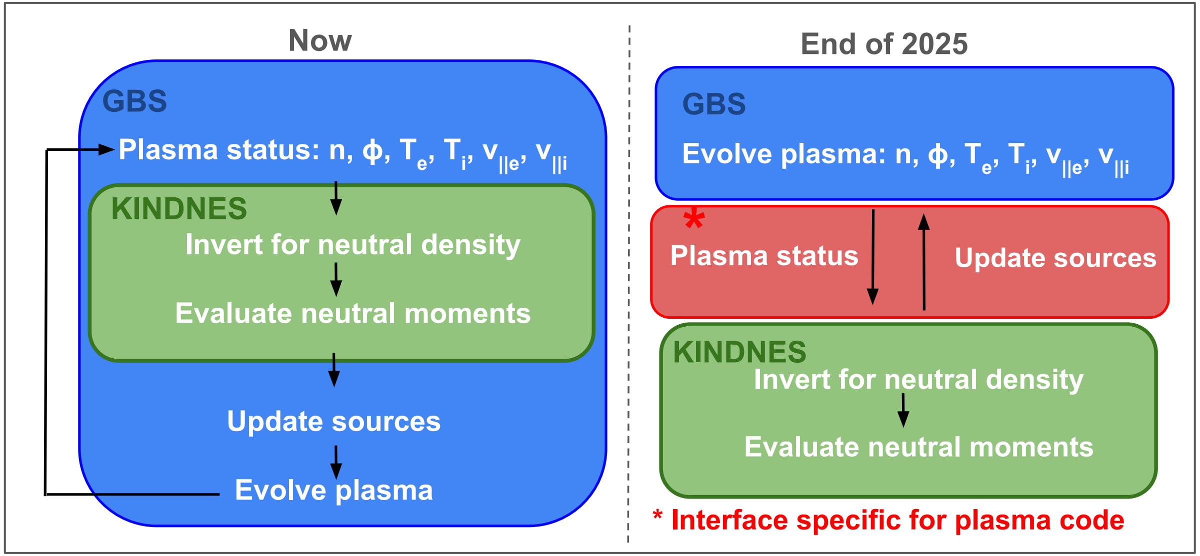


		Time for $\hat{K}_{p\to p}$ [s]		GMRES time [s]		
$N_x \times N_y$	N_P	Dense	HM	Dense	HM	**Memory load**
50×100	5000	24.2	6.8	0.5	0.2	27.2 %
80×160	12800	158.5	20.5	8.2	1.3	12.4 %
100×200	20000	396	33.8	12.1	2	8.4 %
150×300	45000	/	92	/	4.3	4.2 %

[M. Guido et al, submitted to JCP]

On-going: decoupling KINDNES from GBS and GPU implementation

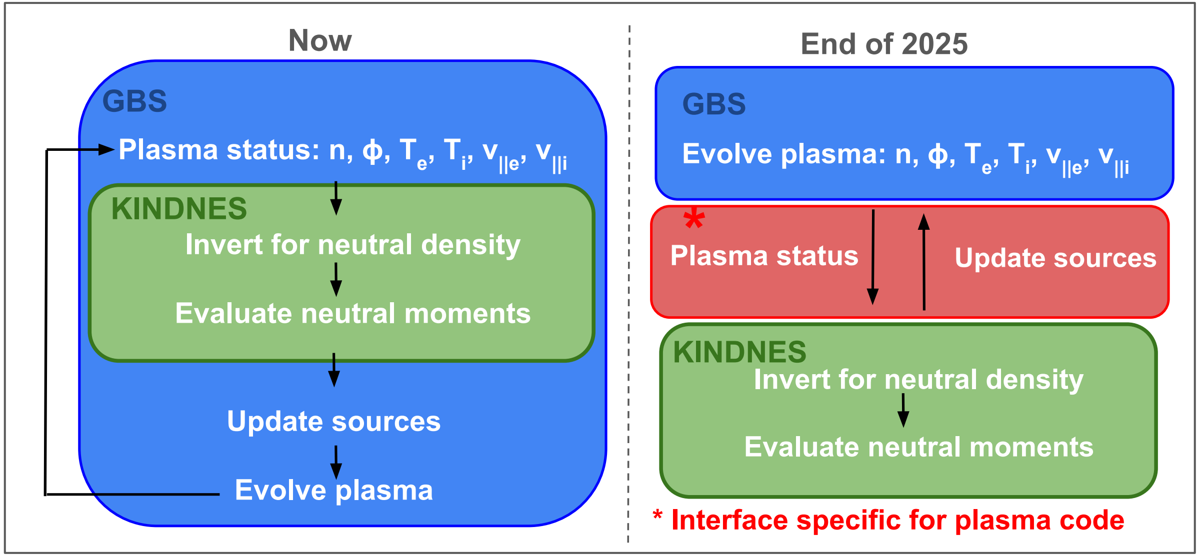
- KINDNES supported by EPFL Advance Computing Hub and applied math groups
- The code runs on all EUROfusion GPU HPC and beyond
- GBS ported on GPU, currently:
    - Plasma evolution on GPU
    - KINDNES on CPU



D. Mancini | TSVV-K | 22-8-2025

On-going: decoupling KINDNES from GBS and GPU implementation

- KINDNES supported by EPFL Advance Computing Hub and applied math groups
- The code runs on all EUROfusion GPU HPC and beyond
- GBS ported on GPU, currently:
  - Plasma evolution on GPU
  - KINDNES on CPU → matrix inversion can be implemented on GPU, interesting with HM



HM Improvement on CPU:

	Time for $\hat{K}_{p\to p}$ [s]	
N_P	Dense	HM
5000	24.2	6.8
12800	158.5	20.5
20000	396	33.8
45000	/	92

Conclusions

KINDNES enables self-consistent **kinetic neutrals** simulations without Monte Carlo noise:

- The model describes **neutral recycling**, reflection and re-emission at the wall, together with all the **most relevant reactions** in the boundary of magnetic fusion devices (ionization, charge-exchange, recombination, dissociation)
- It is possible to extend it with **any** neutral **species**, selecting the relevant reactions (increase of memory usage)
- Implementation highly parallelizable on CPU and portable to GPU
- Code implementation verified and tested
- **Decoupling from GBS ongoing** (end of 2025)

Proposed projects for TSVV-K:

- **Verify** on a test case against Advanced Fluid Neutrals and Kinetic Monte Carlo (put in the context of other methods and codes, see ***D.V. Borodin et al 2022***)
- **Validation** against experimental results
- Further future: extension to additional species

Kernel functions

$$K_{p\to p}^{\mathrm{D,D^+}}(\mathbf{x}_\perp,\mathbf{x}'_\perp) = K_{p\to p,\mathrm{dir}}^{\mathrm{D,D^+}}(\mathbf{x}_\perp,\mathbf{x}'_\perp) + \alpha_{\mathrm{refl}} K_{p\to p,\mathrm{refl}}^{\mathrm{D,D^+}}(\mathbf{x}_\perp,\mathbf{x}'_\perp)$$

$$K_{p\to p,\mathrm{path}}^{\mathrm{D,D^+}}(\mathbf{x}_\perp,\mathbf{x}'_\perp) = \int_0^\infty \frac{1}{r'_\perp}\Phi_{\perp[\mathbf{v}_{\perp,\mathrm{D^+}},T_{\mathrm{D^+}}]}(\mathbf{x}'_\perp,\mathbf{v}_\perp)\exp\left[-\frac{1}{v_\perp}\int_0^{r'_\perp}\nu_{\mathrm{eff,D}}(\mathbf{x}''_\perp)dr''_\perp\right]dv_\perp$$

$$K_{p\to b,\mathrm{path}}^{\mathrm{D,D^+}}(\mathbf{x}_{\perp\mathrm{b}},\mathbf{x}'_\perp) = \int_0^\infty \frac{v_\perp}{r'_\perp}\cos\theta\Phi_{\perp[\mathbf{v}_{\perp,\mathrm{D^+}},T_{\mathrm{D^+}}]}(\mathbf{x}'_\perp,\mathbf{v}_\perp)\exp\left[-\frac{1}{v_\perp}\int_0^{r'_\perp}\nu_{\mathrm{eff,D}}(\mathbf{x}''_\perp)dr''_\perp\right]dv_\perp$$

$$K_{b\to p,\mathrm{path}}^{\mathrm{D,reem}}(\mathbf{x}_\perp,\mathbf{x}'_{\perp\mathrm{b}}) = \int_0^\infty \frac{v_\perp}{r'_\perp}\cos\theta'\chi_{\perp,\mathrm{in,D}}(\mathbf{x}'_{\perp\mathrm{b}},\mathbf{v}_\perp)\exp\left[-\frac{1}{v_\perp}\int_0^{r'_\perp}\nu_{\mathrm{eff,D}}(\mathbf{x}''_\perp)dr''_\perp\right]dv_\perp$$

$$K_{b\to b,\mathrm{path}}^{\mathrm{D,reem}}(\mathbf{x}_{\perp\mathrm{b}},\mathbf{x}'_{\perp\mathrm{b}}) = \int_0^\infty \frac{v_\perp^2}{r'_\perp}\cos\theta\cos\theta'\chi_{\perp,\mathrm{in,D}}(\mathbf{x}'_{\perp\mathrm{b}},\mathbf{v}_\perp)\exp\left[-\frac{1}{v_\perp}\int_0^{r'_\perp}\nu_{\mathrm{eff,D}}(\mathbf{x}''_\perp)dr''_\perp\right]dv_\perp$$

List of Deuterium reactions

Collisional process	Equation	Reaction Frequency
Ionization of D	$e^- + D \rightarrow 2e^- + D^+$	$\nu_{iz,D} = n_e \langle v_e \sigma_{iz,D}(v_e) \rangle$
Recombination of D^+ and e^-	$e^- + D^+ \rightarrow D$	$\nu_{rec,D^+} = n_e \langle v_e \sigma_{rec,D^+}(v_e) \rangle$
$e^- - D$ elastic collisions	$e^- + D \rightarrow e^- + D$	$\nu_{e\text{-}D} = n_e \langle v_e \sigma_{e\text{-}D}(v_e) \rangle$
Ionization of D_2	$e^- + D_2 \rightarrow 2e^- + D_2^+$	$\nu_{iz,D_2} = n_e \langle v_e \sigma_{iz,D_2}(v_e) \rangle$
Recombination of D_2^+ and e^-	$e^- + D_2^+ \rightarrow D_2$	$\nu_{rec,D_2^+} = n_e \left\langle v_e \sigma_{rec,D_2^+}(v_e) \right\rangle$
$e^- - D_2$ elastic collisions	$e^- + D_2 \rightarrow e^- + D_2$	$\nu_{e\text{-}D_2} = n_e \langle v_e \sigma_{e\text{-}D_2}(v_e) \rangle$
Dissociation of D_2	$e^- + D_2 \rightarrow e^- + D + D$	$\nu_{diss,D_2} = n_e \langle v_e \sigma_{diss,D_2}(v_e) \rangle$
Dissociative ionization of D_2	$e^- + D_2 \rightarrow 2e^- + D + D^+$	$\nu_{diss\text{-}iz,D_2} = n_e \langle v_e \sigma_{diss\text{-}iz,D_2}(v_e) \rangle$
Dissociation of D_2^+	$e^- + D_2^+ \rightarrow e^- + D + D^+$	$\nu_{diss,D_2^+} = n_e \left\langle v_e \sigma_{diss,D_2^+}(v_e) \right\rangle$
Dissociative ionization of D_2^+	$e^- + D_2^+ \rightarrow 2e^- + 2D^+$	$\nu_{diss\text{-}iz,D_2^+} = n_e \left\langle v_e \sigma_{diss\text{-}iz,D_2^+}(v_e) \right\rangle$
Dissociative recombination of D_2^+	$e^- + D_2^+ \rightarrow 2D$	$\nu_{diss\text{-}rec,D_2^+} = n_e \left\langle v_e \sigma_{diss\text{-}rec,D_2^+}(v_e) \right\rangle$
Charge-exchange of D^+, D	$D^+ + D \rightarrow D + D^+$	$\nu_{cx,D} = n_{D^+} \langle v_{D^+} \sigma_{cx,D^+}(v_{D^+}) \rangle$
Charge-exchange of D_2^+, D_2	$D_2^+ + D_2 \rightarrow D_2 + D_2^+$	$\nu_{cx,D_2} = n_{D_2^+} \left\langle v_{D_2^+} \sigma_{cx,D_2^+}(v_{D_2^+}) \right\rangle$
Charge-exchange of D_2^+, D	$D_2^+ + D \rightarrow D_2 + D^+$	$\nu_{cx,D\text{-}D_2^+} = n_{D_2^+} \left\langle v_{D_2^+} \sigma_{cx,D\text{-}D_2^+}(v_{D_2^+}) \right\rangle$
Charge-exchange of D_2, D^+	$D_2 + D^+ \rightarrow D_2^+ + D$	$\nu_{cx,D_2-D^+} = n_{D^+} \langle v_{D^+} \sigma_{cx,D_2-D^+}(v_{D^+}) \rangle$

Average always over the faste species: electrons or ions (for cx)