Integral dielectric kernel approach to modelling RF heating in toroidal plasmas: theory and numerical implementation
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Outline

- Introduction to the project:

  Modelling high-frequency heating in tokamaks and stellarators:

  Background, reminder of (mixed FEM-) spectral approaches

  Motivation for our configuration space approach

- Integral kernels in configuration space:

  - Homogeneous plasmas
  - Tokamaks and stellarators

- ENR achievements: theory & numerical implementations, status

- Next steps & prospects

Background

- Waves in hot fusion plasmas: long range wave - particle interactions along equilibrium $\boldsymbol{B_0}$

- Kinetic description of hot plasma HF response in realistic toroidal geometries is challenging: rotational transform, curved $\boldsymbol{B_0}$, $\nabla_{//} B_0 \neq 0$.

- Very different world from $\perp$ - stratified, straight-$\boldsymbol{B_0}$ equilibria

  $\Rightarrow$ Traditional approach to realistic full-wave modelling of wave heating in tokamaks and stellarators relies on Fourier expansions of the HF fields along 2 or 3 spatial coordinates.

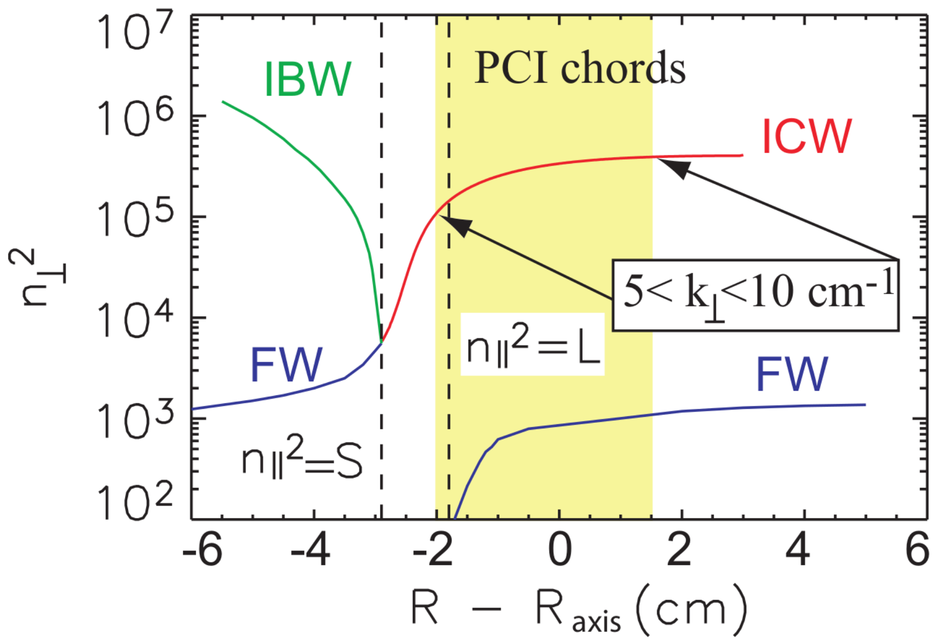
- Indeed allows convenient kinetic theoretical treatment of wave dispersion along curved $\boldsymbol{B_0}$;

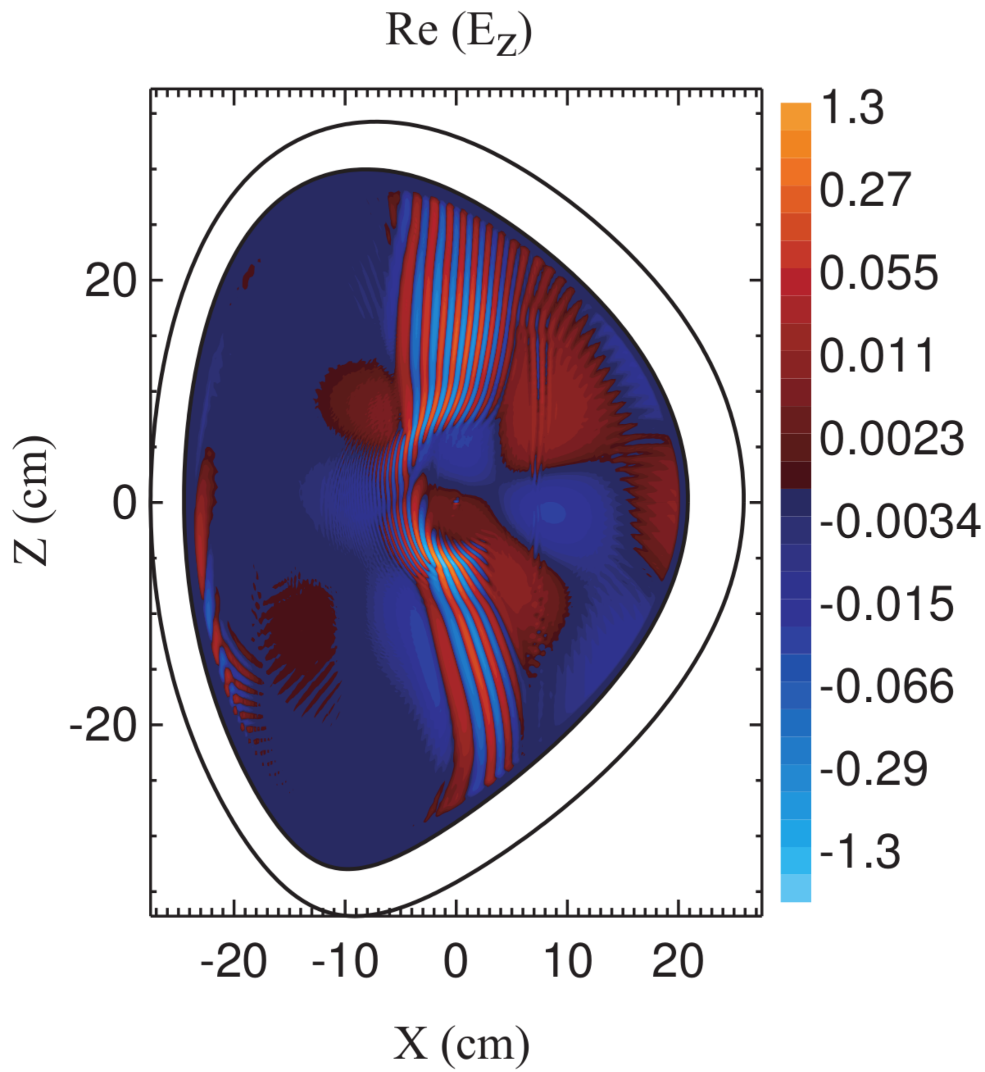
  $\Rightarrow$

  - Mixed spectral-finite element or fully spectral numerical formulations,
  - Plasma described by dielectric tensor formulated in Fourier space involving the well-known $Z^{\{\alpha\}}$ dispersion functions for Maxwellians,
  - Used in most ICRH codes: TORIC, CYRANO, EVE, AORSA, LEMan...

Background: sample advanced ICRF simulations - TORIC

- ICRF, ICW mode conversion in Alcator C-Mod (IBW also excited)
- 240 radial FE, 255 poloidal & 1 toroidal Fourier modes

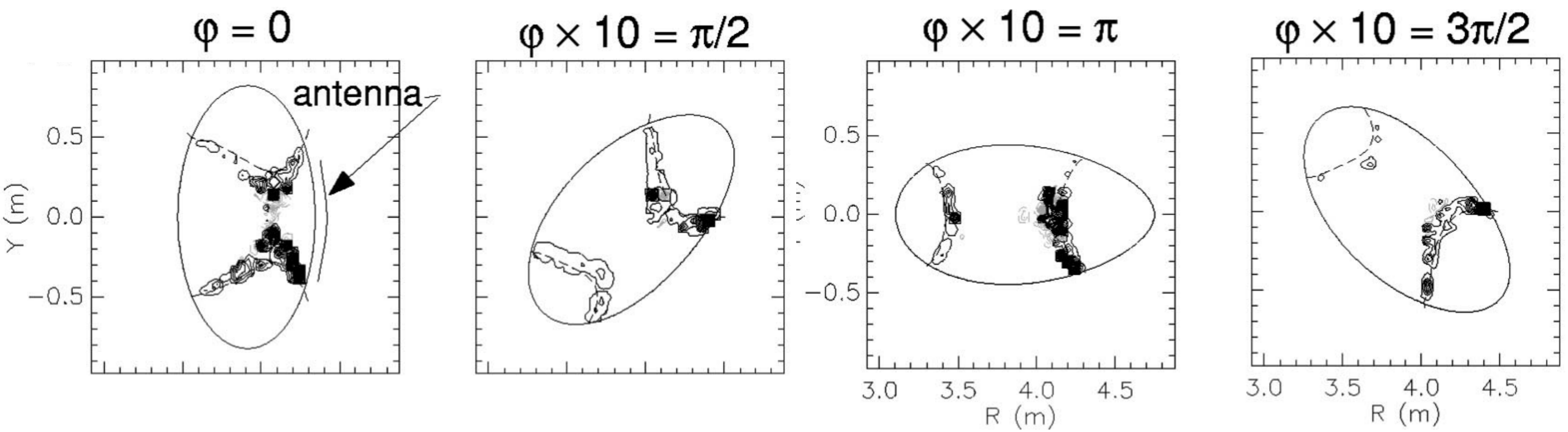




from [Bonoli *et al* 2007]

Background: sample advanced ICRF simulations - AORSA

- ICRF minority ion heating in LHD, 5% H in He
- Fully spectral in cylindrical coordinates, all-orders FLR i.e. arbitrary $k_{\perp}\rho_{LT}$
- 50 modes in R and Y, 16 strongly coupled toroidal modes
- 10 simulations over 1 helical field period allow reconstruction over whole device



Minority heating contours at 4 toroidal positions

from [Jaeger *et al* 2002]

Reference set of equations: linearized Vlasov - Maxwell system

- Maxwell-Vlasov system (frequency domain):

$$\nabla \times \nabla \times \boldsymbol{E} = \left(\frac{\omega}{c}\right)^2 \left(\boldsymbol{I} + \frac{\mathrm{i}}{\omega\,\epsilon_0} \sum_\beta \boldsymbol{\sigma_\beta}\right) . \boldsymbol{E}$$

(**E**: RF electric field)

- RF current density of each species β:

$$\boldsymbol{j}_\beta(\boldsymbol{r}) \equiv \boldsymbol{\sigma}_\beta \cdot \boldsymbol{E} = q_\beta \int f_\beta \boldsymbol{v}\, \mathrm{d}v^3$$

- Vlasov HF perturbed distribution function:

$$f_\beta(\boldsymbol{r}, \boldsymbol{v}) = -\frac{q_\beta}{m_\beta} \int_{-\infty}^{t} \mathrm{e}^{-\mathrm{i}\omega(t'-t)} [\boldsymbol{E}(\boldsymbol{r}') + \boldsymbol{v}' \times \boldsymbol{B}(\boldsymbol{r}')] \cdot \frac{\partial f_{0\beta}}{\partial \boldsymbol{v}'}\, \mathrm{d}t'$$

Reference set of equations: linearized Vlasov - Maxwell system

- Maxwell-Vlasov system (frequency domain), 'weak' form:

$$\frac{\mathrm{i}}{2}\int_{\mathcal{V}}\left[\frac{1}{\omega\,\mu_0}(\nabla\times\boldsymbol{F})^*.(\nabla\times\boldsymbol{E})-\omega\,\varepsilon_0\,\boldsymbol{F}^*.\boldsymbol{E}\right]\,\mathrm{d}r^3+\sum_{\beta}\mathcal{W}_{\boldsymbol{FE}\beta}=-\frac{1}{2}\int_{\mathcal{V}}\boldsymbol{F}^*.\boldsymbol{j}_S\,\mathrm{d}r^3$$

(**E**: RF electric field, **F**: arbitrary test function field, **F** ≡ **E** ⇒ Poynting's theorem)

- This formulation emphasizes the dielectric response of each species β:

$$\boxed{\mathcal{W}_{\boldsymbol{FE}\beta}}=\frac{1}{2}\int_{\mathcal{V}}\boldsymbol{F}^*\cdot\boldsymbol{j}_\beta\,\mathrm{d}r^3=\frac{q_\beta}{2}\int_{\mathcal{V}}\mathrm{d}r^3\int\mathrm{d}v^3\,f_\beta\boldsymbol{F}^*\cdot\boldsymbol{v}.$$

... rather than the RF current density

$$\boldsymbol{j}_\beta(\boldsymbol{r})\equiv\boldsymbol{\sigma}_\beta\cdot\boldsymbol{E}=q_\beta\int f_\beta\boldsymbol{v}\,\mathrm{d}v^3$$

- Vlasov HF perturbed distribution function:

$$f_\beta(\boldsymbol{r},\boldsymbol{v})=-\frac{q_\beta}{m_\beta}\int_{-\infty}^{t}\mathrm{e}^{-\mathrm{i}\omega(t'-t)}[\boldsymbol{E}(\boldsymbol{r}')+\boldsymbol{v}'\times\boldsymbol{B}(\boldsymbol{r}')]\cdot\frac{\partial f_{0\beta}}{\partial\boldsymbol{v}'}\,\mathrm{d}t'$$

- Interests:

  Theory: facilitates consistent treatment of geometry
  Applications: ideally suited for implementation in FEM codes
  & extraction of power balance

Notations

- Field components: left (+) and right (-) circular polarizations, parallel (//)

$$\boldsymbol{E} = E_+\,\boldsymbol{e}_+ + E_-\,\boldsymbol{e}_- + E_{//}\,\boldsymbol{e}_{//} = \sum_{L=-1}^{+1} E_{\mathcal{L}}\,\boldsymbol{e}_{\mathcal{L}}\,, \qquad \boldsymbol{F} = \sum_{L'=-1}^{+1} F_{\mathcal{L}'}\,\boldsymbol{e}_{\mathcal{L}'}$$

$\mathcal{L}$	$+$	$-$	$//$
L	$+1$	-1	0

- '$v_{//}^{\alpha}$ index': $\alpha = \delta_{L,0} + \delta_{L',0}$

$\alpha = 0$: cyclotron and TTMP

$\alpha = 2$: Landau

$\alpha = 1$: mixed Landau-TTMP

- Lowest order FLR: only 'diagonal' contributions $L = L'$, $\alpha = 2\delta_{L,0}$

i.e. the bilinear dielectric response only involves $F_{\mathcal{L}}\,E_{\mathcal{L}}$ terms

- We use p for the cyclotron harmonic index:

$p = +1$: ion fundamental, $p = 0$: Landau, $p = -1$: electron fundamental

Reminder: 3D-spectral stellarator theory (e.g. STELION & LEMan codes)

- Mode expansions:

$$E_{\mathcal{L}} = \sum_{m_1,n_1} E_{m_1,n_1}(\rho)\, e^{i(m_1\theta+n_1\varphi)}, \qquad F_{\mathcal{L}'} = \sum_{m_2,n_2} F_{m_2,n_2}(\rho)\, e^{i(m_2\theta+n_2\varphi)}$$

- Strong coupling between poloidal and toroidal modes in plasma response: to lowest order FLR,

$$\mathcal{W}_{\boldsymbol{FE}\beta} = \sum_{p=-1}^{+1} \sum_{m_1,\, m_2, n_1, n_2 = -\infty}^{+\infty} \mathcal{W}^{p}_{21\beta}$$

Matrix elements: Fourier transforms of Maxwellian dispersion functions $Z^{\{\alpha\}}$

$$\mathcal{W}^{p}_{21\beta} = -\mathrm{i}\pi\varepsilon_0 \sum_{L=-1}^{+1} \delta_{L,p} \int \mathrm{d}\rho\, F^{*}_{\mathcal{L},m_2,n_2} \left\{ \iint \mathcal{J}\, \frac{2^{\alpha/2}\,\omega_{\mathrm{p}}^2}{\left|k_{//\bar{m},\bar{n}}\right| v_{\mathrm{T}}} Z^{\{\alpha\}}\!\left(\frac{\omega - p\omega_{\mathrm{c}}}{\left|k_{//\bar{m},\bar{n}}\right| v_T} \right) \mathrm{e}^{\mathrm{i}[(m_1-m_2)\theta + (n_1-n_2)\varphi]}\, \mathrm{d}\theta\, \mathrm{d}\varphi \right\} E_{\mathcal{L},m_1,n_1}$$

$(\alpha = 2\delta_{L,0})$

- The // wavenumber varies with position:

- with standard angle variables, $k_{//}(\rho,\theta,\varphi) = \bar{m}\sin\Theta/N_\theta + \bar{n}\cos\Theta/R$

- with the straight-magnetic-field angles $(\bar{\theta},\bar{\varphi})$ of [Lamalle 2006],

$$k_{//}(\rho) = (\bar{m} + \bar{n}q)/H(\rho)$$

- Hamiltonian and fully consistent gc theories lead to the symmetric indices

$\boxed{\bar{m} = (m_1+m_2)/2,\ \bar{n} = (n_1+n_2)/2}$, $\Rightarrow$ purely resonant wave absorption

(Tokamak: CYRANO and EVE codes [Lamalle 1997, Dumont 2009]; stellarator: unpublished)

Now, to the point: Integral kernels in configuration space

- Focus of our ENR: alternative approach, with the plasma dielectric response formulated as a nonlocal integral operator in physical space, involving new Maxwellian '*kernel dispersion functions (KDF)*'.

- Max. analytical developments
  $\Rightarrow$ extra physics insight and faster numerical simulations.

- Current emphasis on long-range dispersion effects along $\boldsymbol{B_0}$, outstanding in applications. Implementation in progress.

- N.B. For the sake of clarity, presentation only shows lowest order FLR.

Advantages of the configuration space integral approach

- Leaves complete freedom of choice for numerical discretization.

- $\Rightarrow$ Enables FEM methods in 2D and 3D to model wave propagation and absorption in hot inhomogeneous fusion plasmas;

- Enables local mesh refinements (ruled out with spectral methods), essential to address FLR effects in 2D / 3D;

- Better suited field representations to deal with FLR in toroidal geometry;

- Straightforward connection with RF antenna models based on the FEM.

$\Rightarrow$ Main goals of the project:

- Efficient implementation in new full-wave code & existing FEM packages;

- Validation, demonstration of attractiveness, model RF heating in tokamaks and stellarators.

Earlier work on the configuration space approach

- Sauter & Vaclavik 1992, 1994, Smithe *et al* 1997: $\perp$-stratified plasma, focus on $\perp$ nonlocal effects, spectral in // direction

- Meneghini, Shiraiwa & Parker 2009: LHCD, integral treatment of Landau damping, iterative solution

- Svidzinski 2016: very general approach, hot conductivity kernel evaluated numerically by orbit integration

- Fukuyama, 2019 RFPPC

- Lamalle, 2019 RFPPC, 2023 EFTC, 2024 Varenna: tokamak theory, // treatment

- Machielsen, Rubin & Graves 2023: full FLR theory for homogenous plasmas, both // and $\perp$ treatments. Applied to $\perp$ but not yet to //.

Other recent treatments of // dispersion by iterative methods:
Vallejos *et al* 2018, Zaar *et al* 2024

// - homogeneous plasmas: comparison spectral / configuration space

- In common: Maxwell-Vlasov system, weak form:

$$\frac{\mathrm{i}}{2}\int_{\mathcal{V}}\left[\frac{1}{\omega\,\mu_0}(\nabla\times\boldsymbol{F})^*.(\nabla\times\boldsymbol{E})-\omega\,\varepsilon_0\,\boldsymbol{F}^*.\boldsymbol{E}\right]\mathrm{d}r^3+\sum_{\beta}\mathcal{W}_{\boldsymbol{FE}\beta}=-\frac{1}{2}\int_{\mathcal{V}}\boldsymbol{F}^*.\,\boldsymbol{j}_S\,\mathrm{d}r^3$$

- Plasma response, 0th order FLR, showing homogeneous plasma case,
  - Usual spectral formulation: local in $k_{//}$ space

$$\mathcal{W}_{\boldsymbol{FE}\beta}=-\mathrm{i}\pi\varepsilon_0\sum_{L=-1}^{1}\delta_{L,p}\,2^{\alpha/2}\int\mathrm{d}r_\perp^2$$

$$\int_{-\infty}^{+\infty}\mathrm{d}k_{//}\;\tilde{F}^*_{\mathcal{L}}(k_{//})\,\frac{\omega_\mathrm{p}^2}{k_{//}v_\mathrm{T}}\boxed{Z^{\{\alpha\}}\left(\frac{\omega-p\omega_\mathrm{c}}{k_{//}v_\mathrm{T}}\right)}\,\tilde{E}_{\mathcal{L}}(k_{//})\qquad(\alpha=2\delta_{L,0})$$

  - Configuration space formulation: obtained by back Fourier transform of $\tilde{E}_{\mathcal{L}}(k_{//})$

$$\mathcal{W}_{\boldsymbol{FE}\beta}=-\frac{\mathrm{i}\varepsilon_0}{2}\sum_{L=-1}^{1}\delta_{L,p}\,2^{\alpha/2}\int\mathrm{d}r_\perp^2$$

$$\int\mathrm{d}z\int\mathrm{d}z'\;F^*_{\mathcal{L}}(z')\,\frac{\omega_\mathrm{p}^2}{v_\mathrm{T}}\boxed{\Upsilon_\alpha\left(\frac{\omega-p\,\omega_\mathrm{c}}{2\,v_\mathrm{T}}\,|z'-z|\right)}\,E_{\mathcal{L}}(z)\qquad(\alpha=2\delta_{L,0})$$

Configuration space approach: infinite //-homogeneous plasmas

- Plasma response, showing 0th order FLR: involves nonlocal integrals (z, z') along magnetic field lines

$$\mathcal{W}_{\boldsymbol{FE}\beta} = -\frac{\mathrm{i}\varepsilon_0}{2} \sum_{L=-1}^{1} \delta_{L,p}\, 2^{\alpha/2} \int \mathrm{d}r_\perp^2\, \frac{\omega_\mathrm{p}^2}{v_\mathrm{T}} \int \mathrm{d}z \int \mathrm{d}z'\; F_\mathcal{L}^*(z')\, \boxed{\Upsilon_\alpha(\lambda\,|z'-z|)}\; E_\mathcal{L}(z) \quad (\alpha = 2\delta_{L,0})$$

$$\lambda = \frac{\omega - p\,\omega_\mathrm{c}}{2\,v_\mathrm{T}}$$

The kernel dispersion functions (KDF) are derived from the usual PDF:

$$\boxed{\Upsilon_\alpha(\xi) = \frac{1}{\pi}\int_0^{+\infty} Z^{\{\alpha\}}\left(\frac{2\,\xi}{t}\right) \left(\begin{array}{c} \cos t \\ \mathrm{i}\,\sin t \end{array}\right) \frac{\mathrm{d}t}{t}, \qquad \alpha \left\{\begin{array}{c} \text{even} \\ \text{odd} \end{array}\right\}, \qquad \mathrm{Im}\,\xi > 0}$$

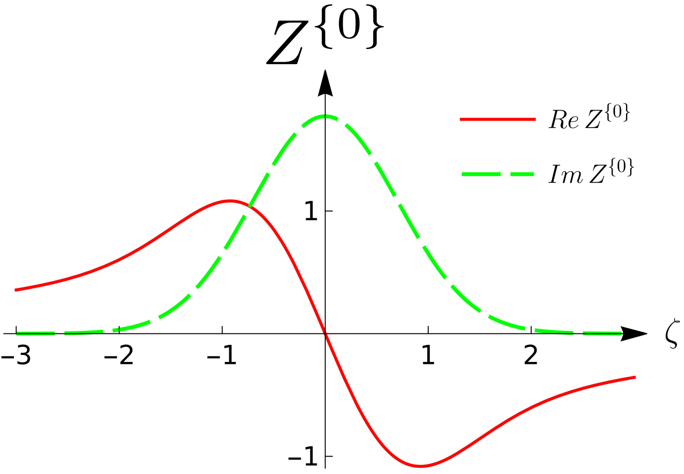
- $Z^{\{\alpha\}}$: standard plasma dispersion functions (PDF)
- Odd α kept for completeness, $\alpha = 1$ enters the FLR theory
- The form of Υ_0 appears in Svidzinski (2016)'s conductivity kernel.

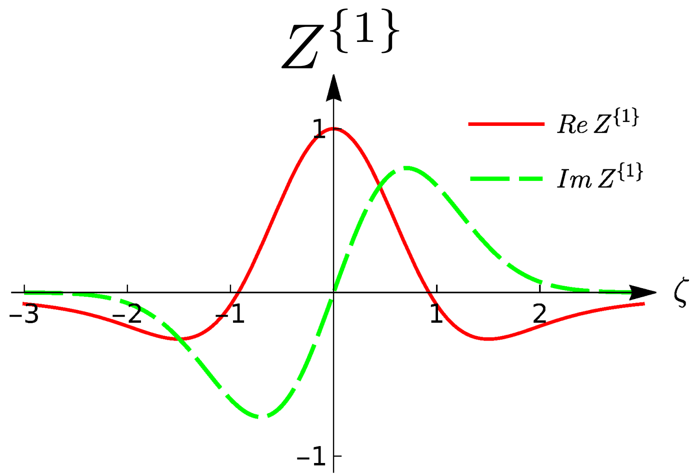
  Lamalle (2023, 2024), equivalent to Machielsen *et al* (2023) S_1, S_2, S_3 but ≠ def.

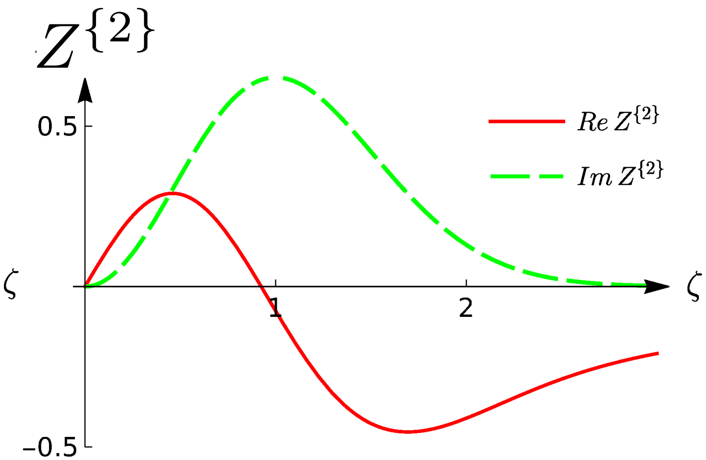
[Lamalle *et al*, RFPPC2025]

Correspondence

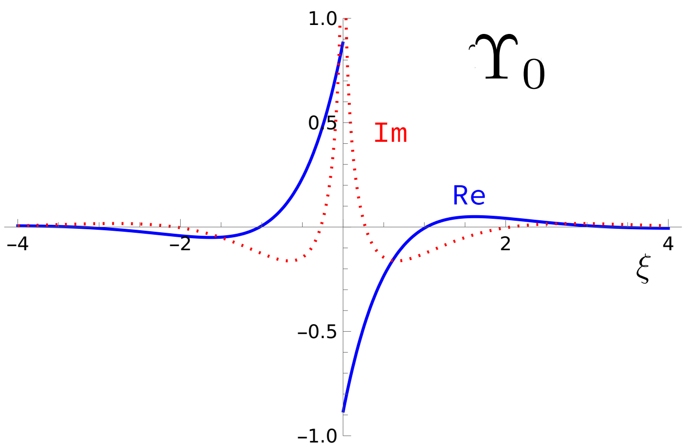
- Plasma dispersion functions: at $k_{//} = 0$, $\mathrm{Im}\,Z^{\{\alpha\}}(\pm\infty) = 0$

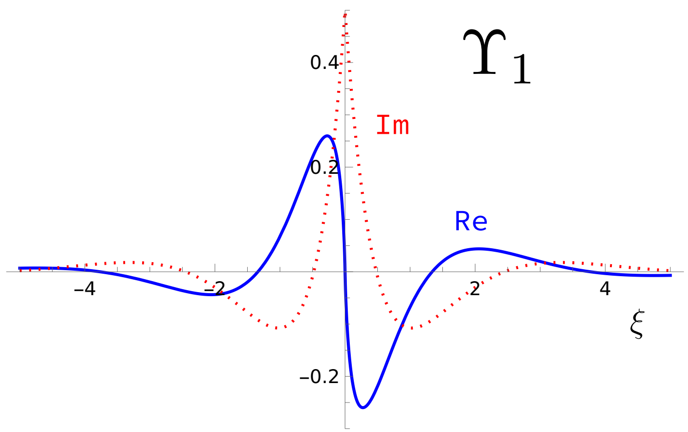


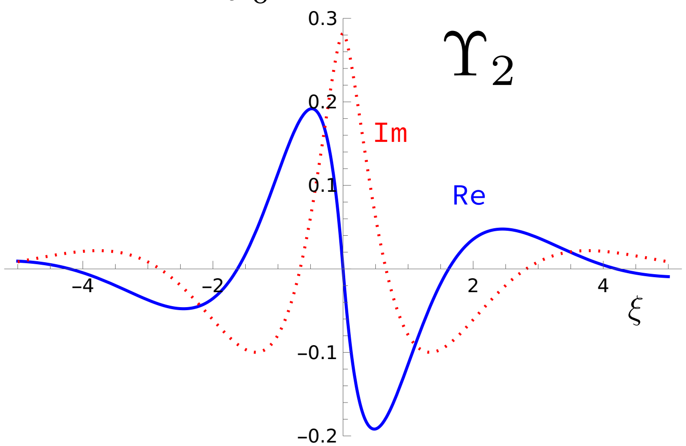




- Kernel dispersion functions: on uniform RF field, $\int_0^\infty \mathrm{Im}\,\Upsilon_\alpha(\xi)\,\mathrm{d}\,\xi = 0$







Cyclotron (p≠0), TTMP
Im log singular at $\xi = 0$

Mixed TTMP-Landau

Landau (p=0)

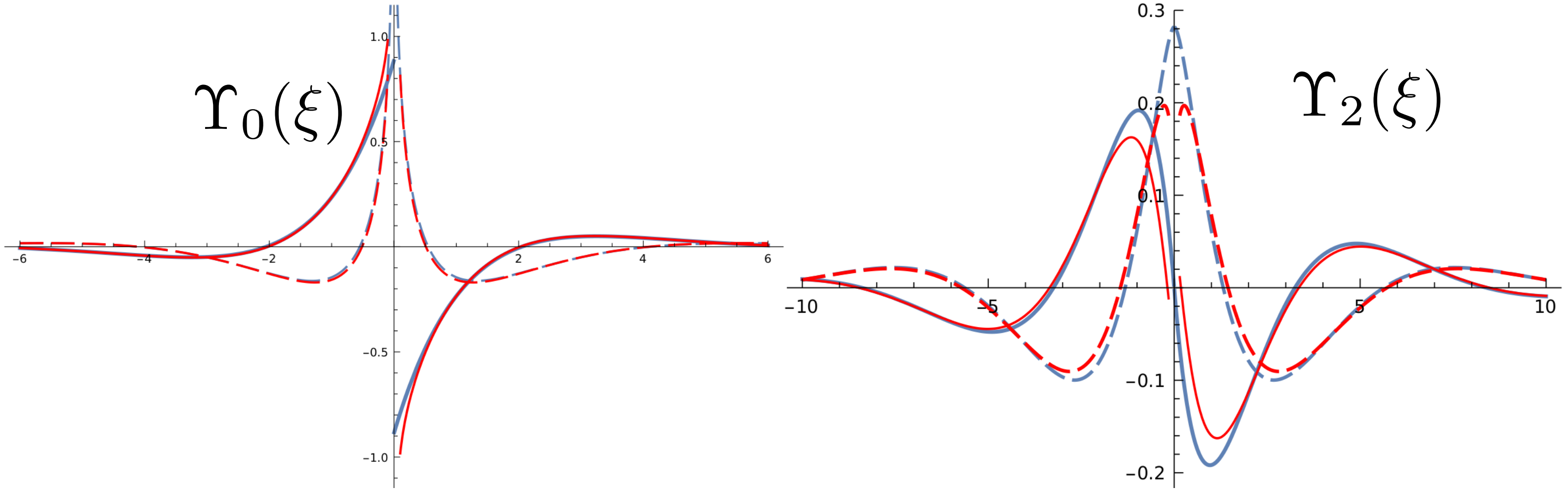
Detailed analytical study of the KDF

- Definition ensures genuine functions of a single variable
- Many analytical properties established and exploited in the applications:
  Series expansions, integral representation, recurrences, differential equation, ...

$$\Upsilon_\alpha(\xi) = \frac{\mathrm{i}}{2\sqrt{\pi}} \int_{\to 0}^{+\infty} \mathrm{e}^{-u+2\mathrm{i}\xi/\sqrt{u}}\, u^{\alpha/2-1}\, \mathrm{d}u\,, \quad \mathrm{Im}\,\xi > 0$$

- Leading asymptotic expansion emphasizes their compact support:

$$|\Upsilon_\alpha(\xi)| \sim \exp\{-3\,|\xi|^{2/3}/2\}\; |\xi|^{(\alpha-1)/3}/\sqrt{3}, \quad \xi \to \pm\infty$$



Blue: series,
Red: asymptotic exp.

- Efficient numerical evaluation by complementary methods ⇒ tabulations
  Our FE codes interpolate in persistent tables.

[Lamalle *et al*, RFPPC2025]

Detailed analytical study of the KDF

Latest analytical results (2025):

- Thanks to the differential relation $\boxed{\Upsilon'_{\alpha}(\xi) = 2\mathrm{i}\,\Upsilon_{\alpha-1}(\xi)}$

  the following double // integrals are evaluated analytically in terms of the same family of special functions:

$$(z_2 - z_1)(z'_2 - z'_1)\, D_{\alpha}^{(m,n)} = \int_{z'_1}^{z'_2} \mathrm{d}z' \left(\frac{z' - z'_1}{z'_2 - z'_1}\right)^m \int_{z_1}^{z_2} \mathrm{d}z\, \Upsilon_{\alpha}(\beta\,|z - z'|) \left(\frac{z - z_1}{z_2 - z_1}\right)^n$$

- Thanks to the recurrence relation $\boxed{\Upsilon_{\alpha+2}(\xi) = \frac{\alpha}{2}\,\Upsilon_{\alpha}(\xi) - \mathrm{i}\xi\,\Upsilon_{\alpha-1}(\xi)}$

  a KDF of arbitrary index can be evaluated in terms of $\Upsilon_0, \Upsilon_1, \Upsilon_2$ alone.

- The nonlocal interactions between FE (or B-spline) basis functions can be evaluated semi-analytically, using the $D_{\alpha}^{(m,n)}$ as building blocks.

  $\Rightarrow$ Code simplification and strong acceleration!

- Detailed theory paper on the KDF soon to be submitted for publication.

‘Lumping’ the FEM system matrix

$$\Upsilon_\alpha\left(\frac{\omega - p\,\omega_\mathrm{c}}{v_\mathrm{T}}\,\frac{|z'-z|}{2}\right)$$

- // - homogeneous plasma: significant // nonlocal cyclotron interaction $(\alpha = 0)$ over

$$|z'-z| \lesssim \frac{K\,v_\mathrm{T}}{\omega - p\,\omega_\mathrm{c}}, \qquad K \sim \text{"}8\text{"}$$

Careful treatment at resonance layer...

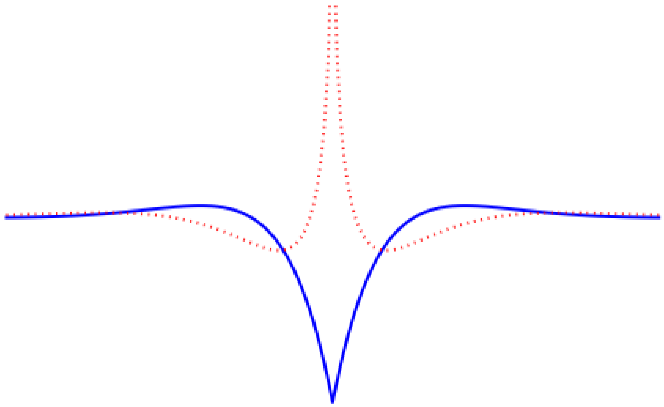
- Nonlocal Landau interaction $(\alpha = 2)$ over

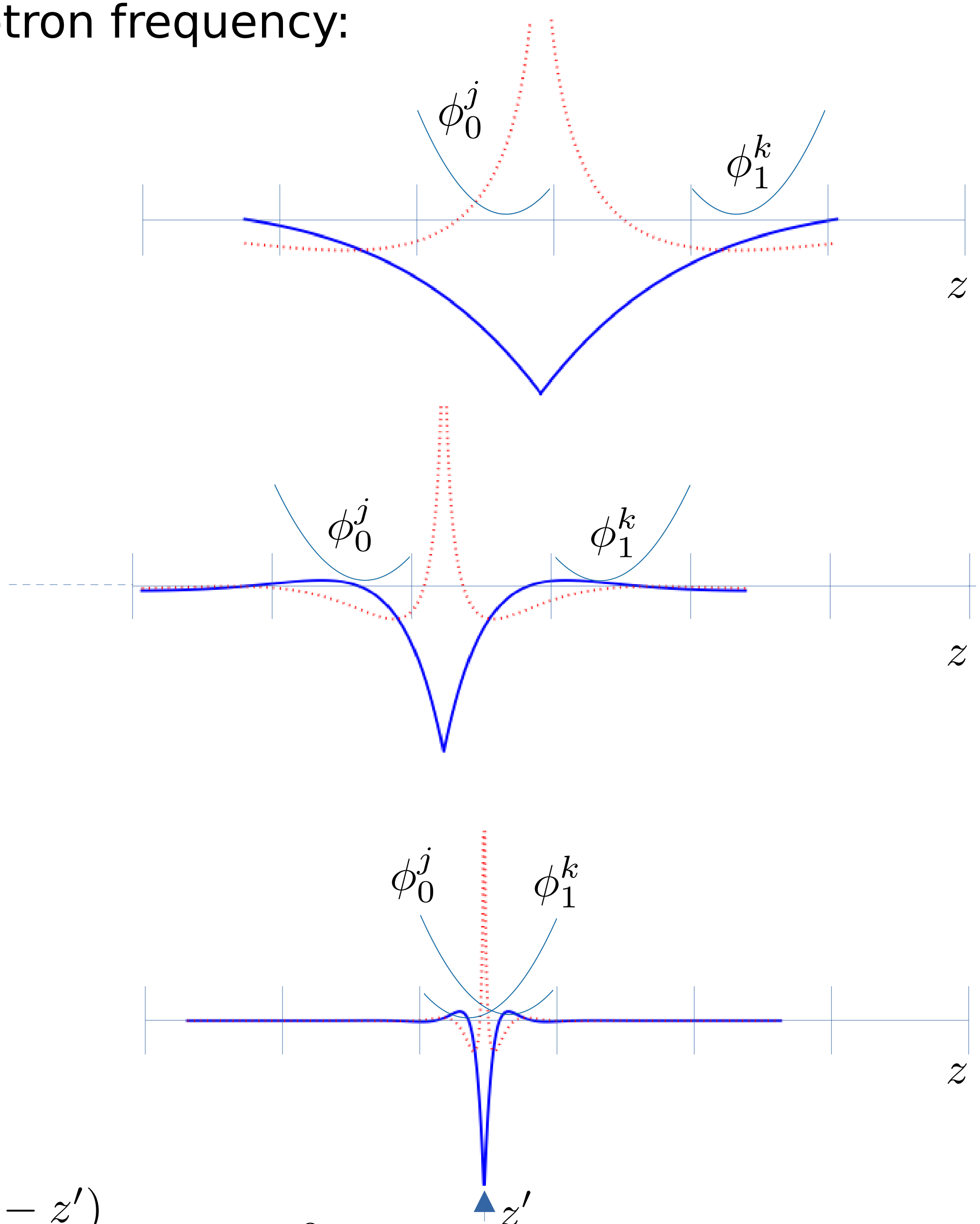
$$|z'-z| \lesssim K\,\frac{v_\mathrm{T}}{\omega}, \qquad K \sim \text{"}8\text{"}$$

Cyclotron interaction - FEM numerical treatment

Decreasing values of T_i at fixed z' and cyclotron frequency:

$$\Upsilon_0 \text{ vs } \left|z - z'\right| :$$





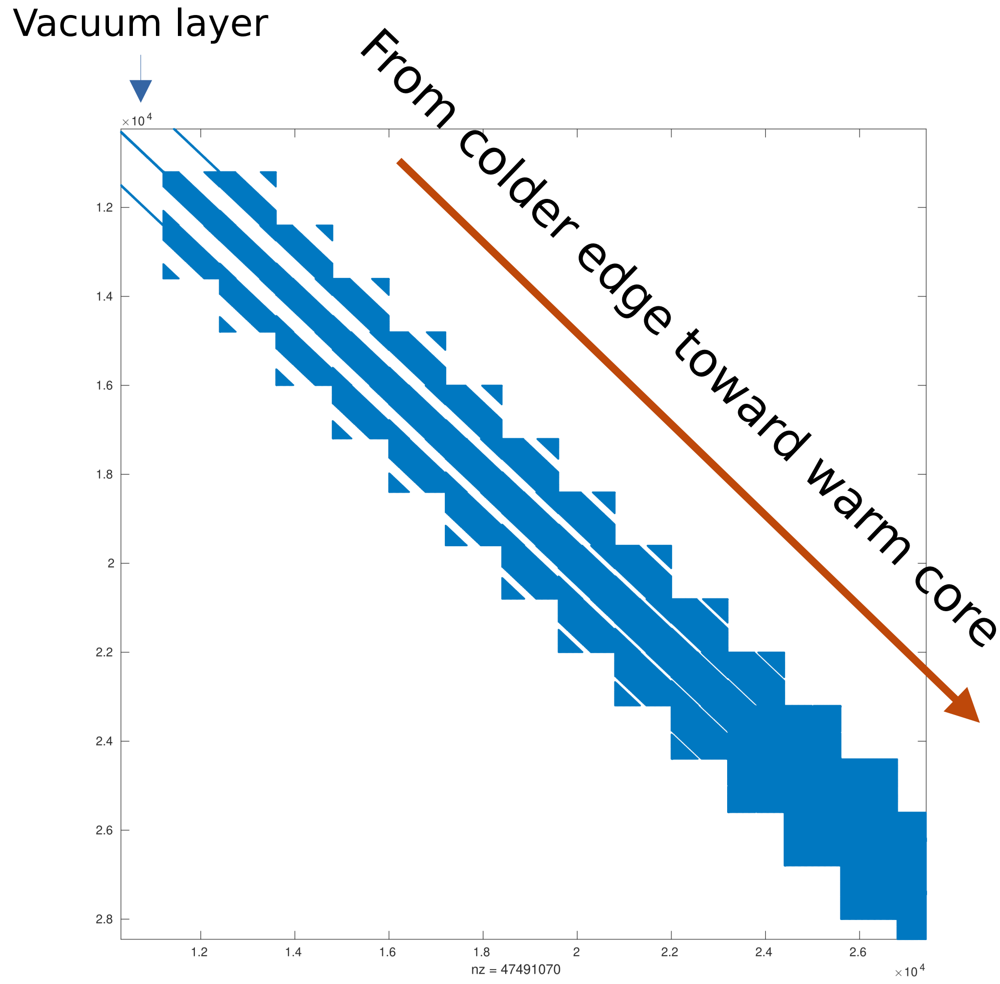
Cold limit: $v_T^{-1}\, \Upsilon_0 \left(\frac{\omega - p\,\omega_c}{2\, v_T} \left|z - z'\right| \right) \sim -\frac{\delta(z - z')}{\omega - p\omega_c}, \quad v_T \to 0$

FEM global matrix: sparsity pattern on 2D slab code

Minority H in D

Element numbering sequential along field lines, then across

Domain periodic in // direction



[Reman *et al*, RFPPC2025]

Tokamak: full configuration space result

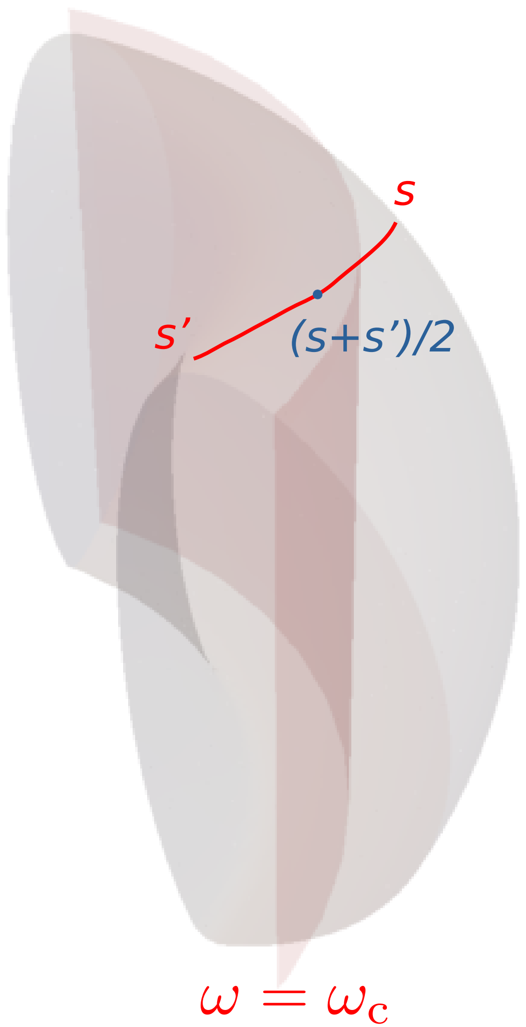
- Analytical developments remove the poloidal & toroidal Fourier expansions.
- Plasma response result, showing 0th order FLR: involves nonlocal integrals (*s*, *s'*) along magnetic field lines,

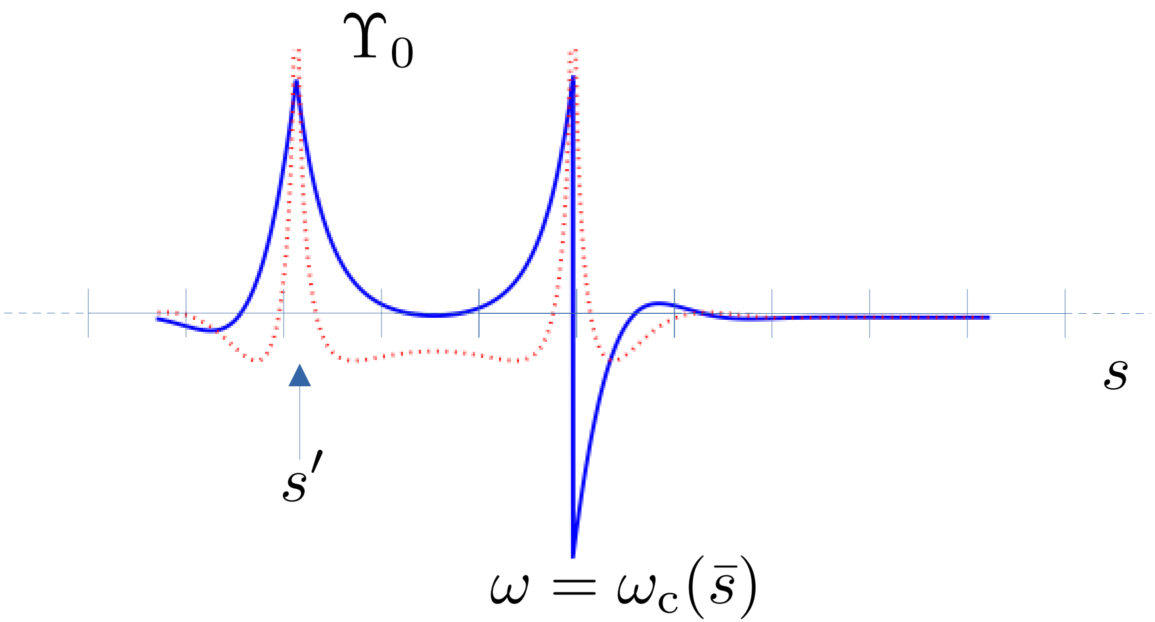
$$\mathcal{W}_{\boldsymbol{F}\boldsymbol{E}\beta} = -\frac{\mathrm{i}\varepsilon_0}{2} \sum_{L=-1}^{1} \delta_{L,p}\, 2^{\alpha/2} \int \mathrm{d}r_\perp^2\, \frac{\omega_\mathrm{p}^2}{v_\mathrm{T}} \int \mathrm{d}s \int \mathrm{d}s'\, F_\mathcal{L}^*(s')\, \boxed{\Upsilon_\alpha \left[\hat{\xi}_p(s,s')\right]}\, E_\mathcal{L}(s) \qquad (\alpha = 2\delta_{L,0})$$

- These are the same KDFs as for infinite homogeneous plasmas!
- Here, their argument is evaluated at the mid-point between field (*s*) and test (*s'*) points:

$$\boxed{\hat{\xi}_p(s,s') = \frac{\omega - p\omega_\mathrm{c}\left(\frac{s+s'}{2}\right)}{v_\mathrm{T}}\, \frac{|s'-s|}{2}}$$

The cyclotron KDF in presence of $\nabla_{//} B_0$



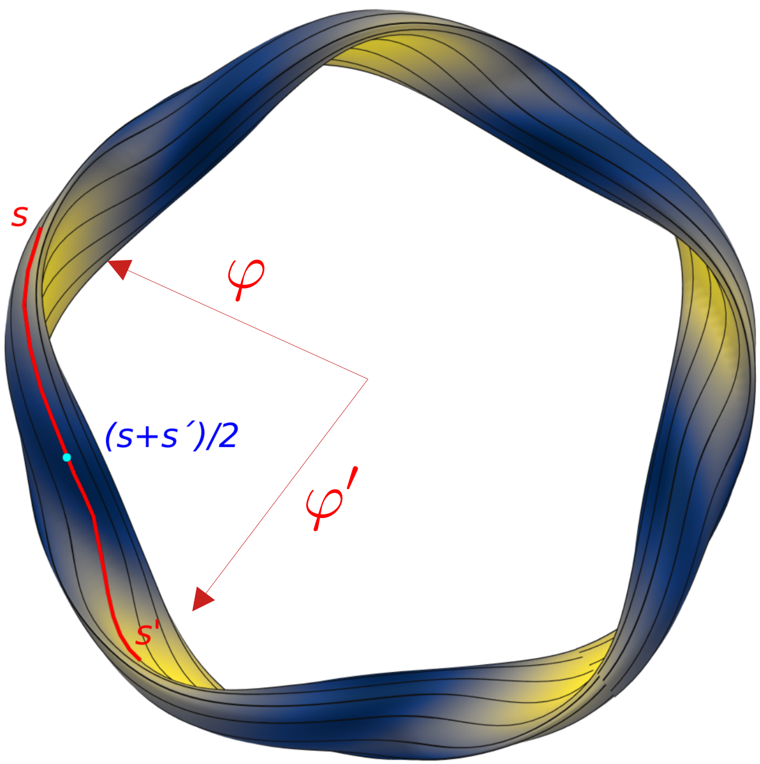


Stellarator: configuration space result

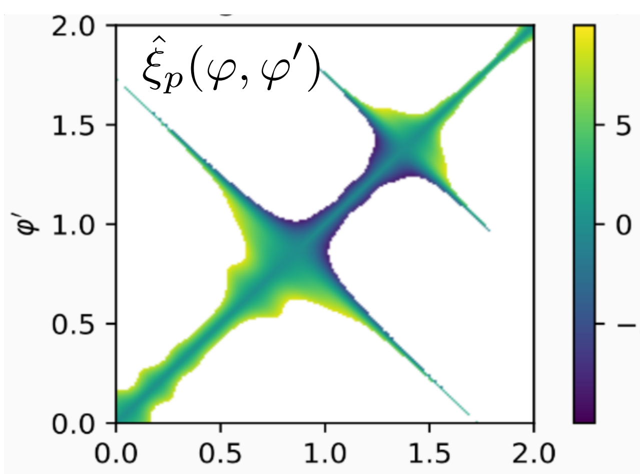
- Same method applied to spectral Maxwellian plasma response of [Vdovin 1996, Fukuyama 2000, Murakami 2006]* $\Rightarrow$ same formal expression as for tokamaks:

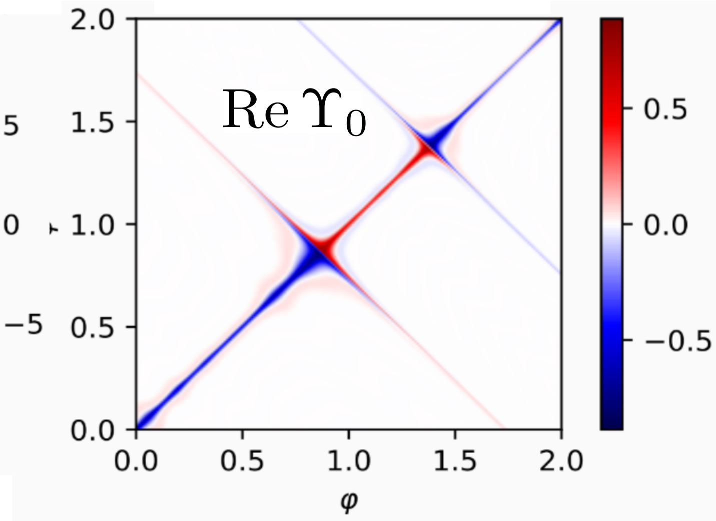
$$\mathcal{W}_{\boldsymbol{FE}\beta} = -\frac{\mathrm{i}\varepsilon_0}{2} \sum_{L=-1}^{1} \delta_{L,p}\, 2^{\alpha/2} \int \mathrm{d}r_\perp^2\, \frac{\omega_\mathrm{p}^2}{v_\mathrm{T}} \int \mathrm{d}s \int \mathrm{d}s'\; F_\mathcal{L}^*(s')\; \Upsilon_\alpha \left[\hat{\xi}_p(s,s')\right]\; E_\mathcal{L}(s) \qquad (\alpha = 2\delta_{L,0})$$

$$\hat{\xi}_p(s,s') = \frac{\omega - p\omega_\mathrm{c}\left(\frac{s+s'}{2}\right)}{v_\mathrm{T}}\; \frac{|s'-s|}{2}$$



Cyclotron kernel variation along sample field line:





- Interaction mostly involves close neighbour points, except near 2 cyclotron layers
- Helps determine system sparsity pattern for 3D stellarator geometry

* Ignoring drift waves and specific $\partial f_0/\partial\psi$ effects; and assuming integrable orbits.

Progressive FEM implementation

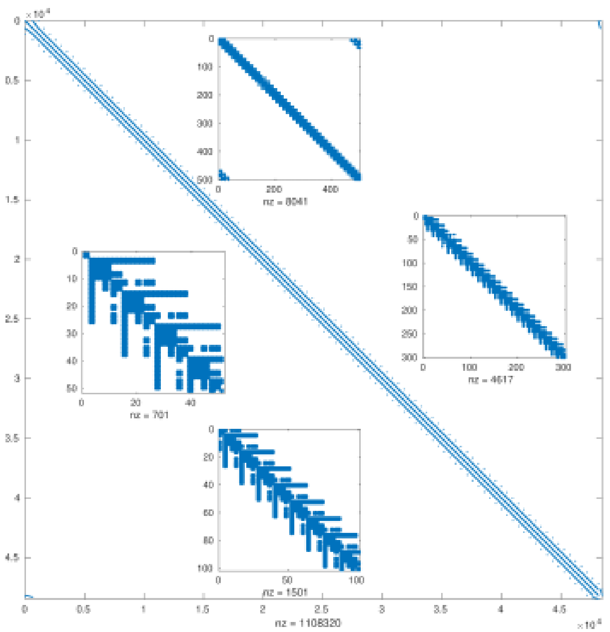
- Following three paths:
  - In-house PLIKES code, 2.5D slab model (quadratic Nédélec+Lagrange)
  - NMPP Garching's Psydac (tensor product B-splines): implementation under way.
  - ULiège's Gmsh-FEM (high degree polynomials): in progress,
    PhD started in October 2024,
    Specific goal: enabling / optimizing very large scale computing.
    Linear system preconditioning, domain decomposition and iterative methods
    (innovative for Maxwell's equations).

- Staged development, initially FLR^0 $\Rightarrow$ minority & 3-ion ICRH scenarios

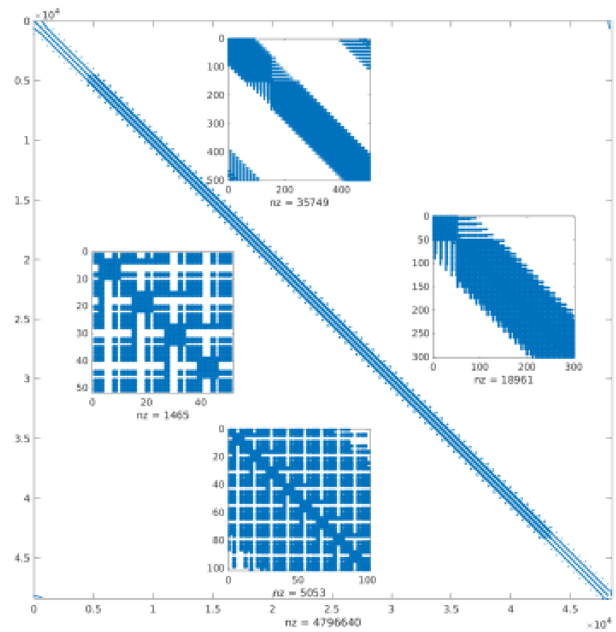
[Reman *et al*, RFPPC2025]

PLIKES 2.5D slab code

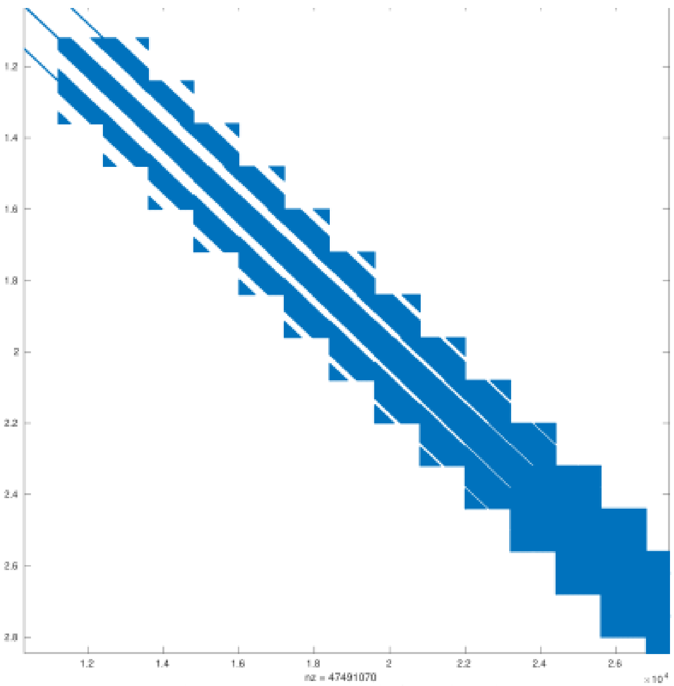
- Written in C, uses openMP, MUMPS, hdf5 outputs,
- Curl-conforming finite elements $\Rightarrow$ stable discretization of Maxwell operator
- DOFs labelled to follow $\parallel$ nonlocality
- Gauss and Log numerical quadratures (singular cyclotron kernel)
  … being replaced by semi-analytical expressions
- Kernel cuff-off defining non local-coupling range



$T_H = 0\text{keV}$, homogeneous plasma



$T_H = 4\text{keV}$, homogeneous plasma



Sparsity, $\perp$-stratified plasma

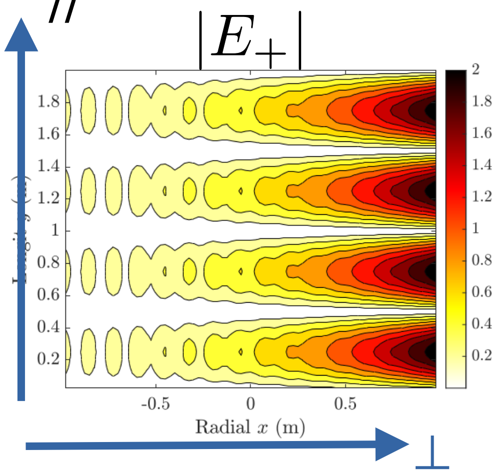
[Reman *et al*, RFPPC2025]

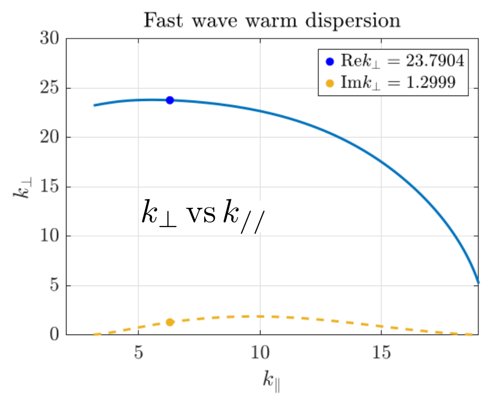
PLIKES 2.5D slab code, integral kernel benchmarks

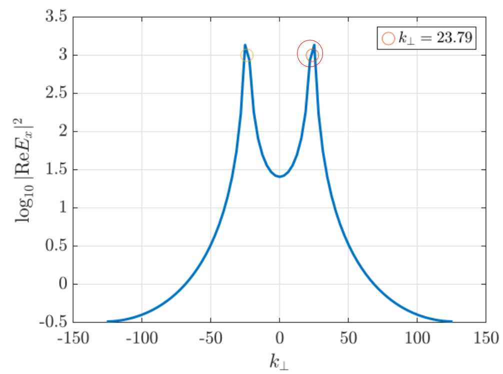
Sine-wave antenna current (51.8MHz, $k_{//}$= 2π m^{-1})

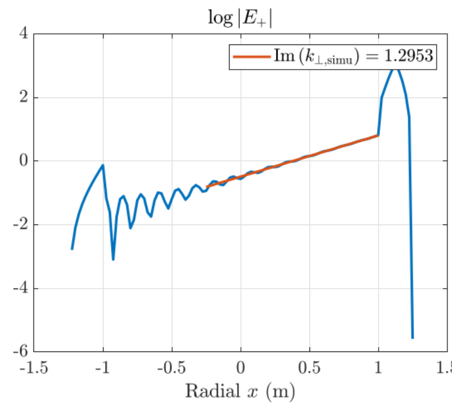
Homogeneous plasma:

Solution matches Re & Im $k_\perp$:



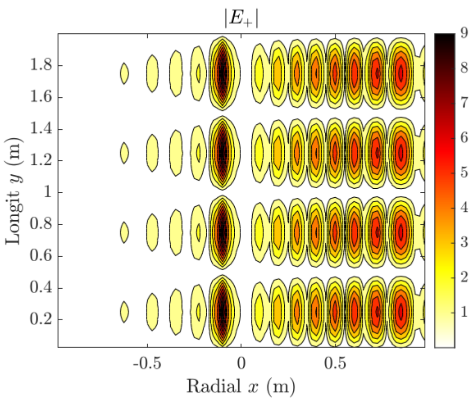


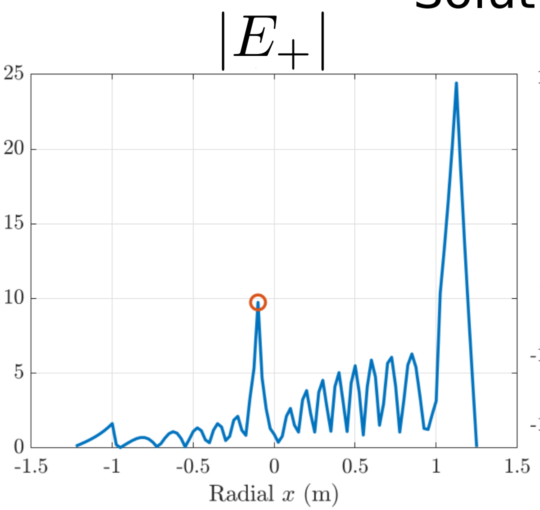


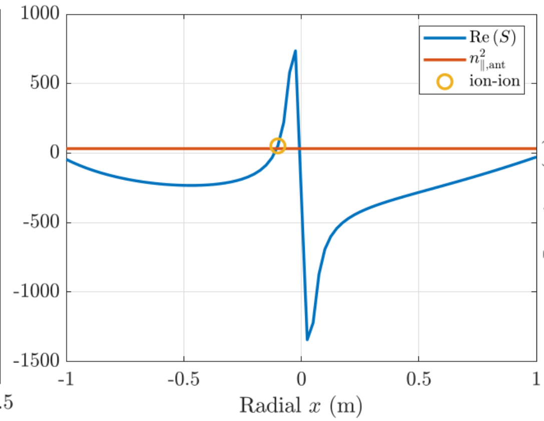


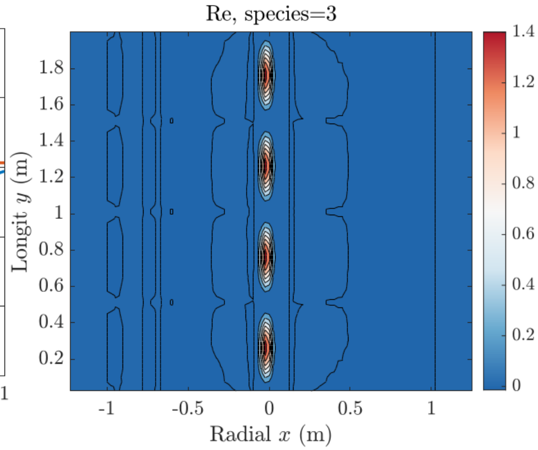
⊥-stratified plasma:

Solution matches ion-ion hybrid layer





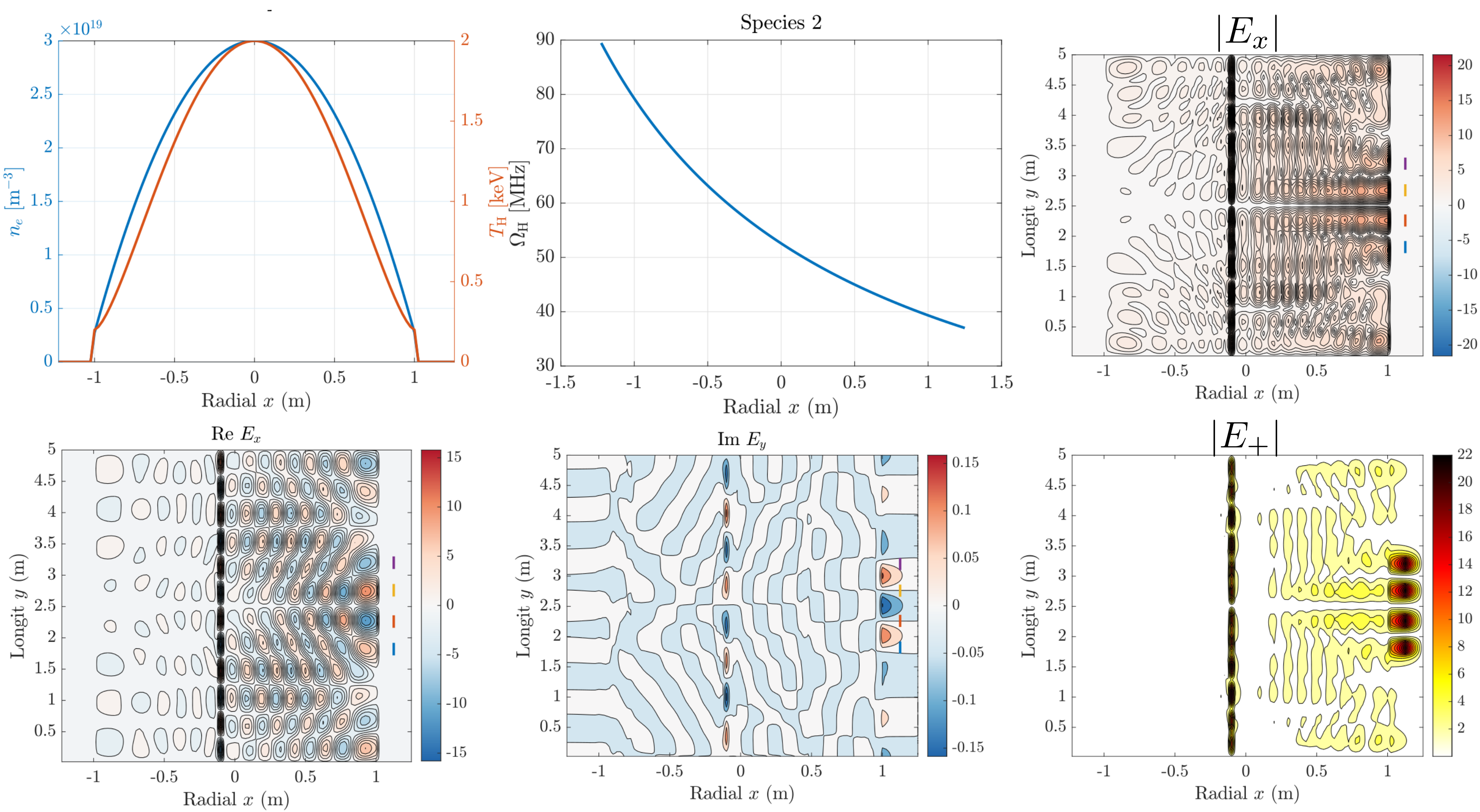




[Reman *et al*, RFPPC2025]

PLIKES 2.5D slab code: first results with integral kernel

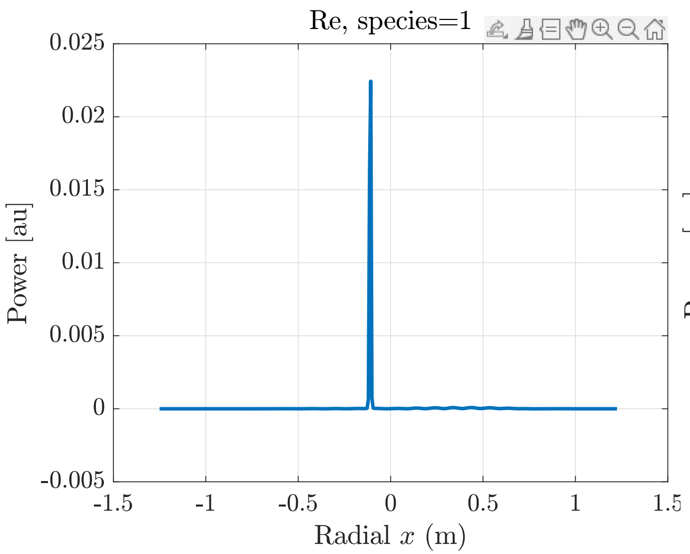
- JET parameters: 3.45T, 3x10^{19}m^{-3}, 2keV, 5% H in D, A2 antenna (0π0π), 52.6MHz
- Four-strap antenna // spectrum captured in a single FE simulation



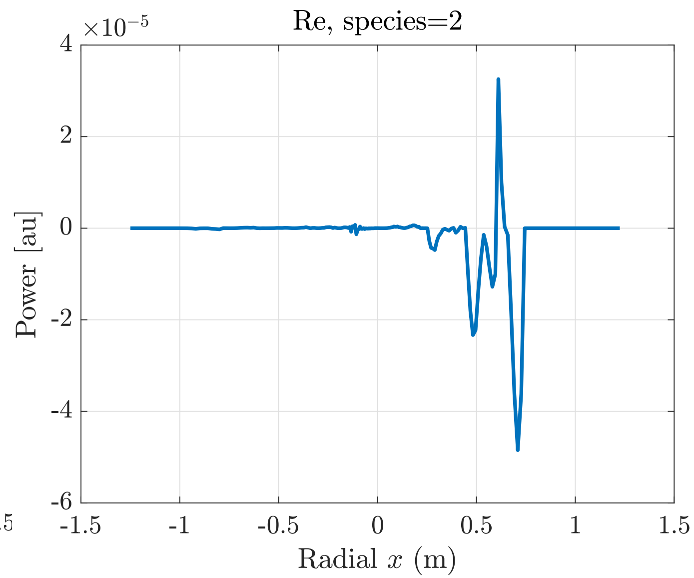
[Reman *et al*, RFPPC2025]

PLIKES 2.5D slab code: first results with integral kernel

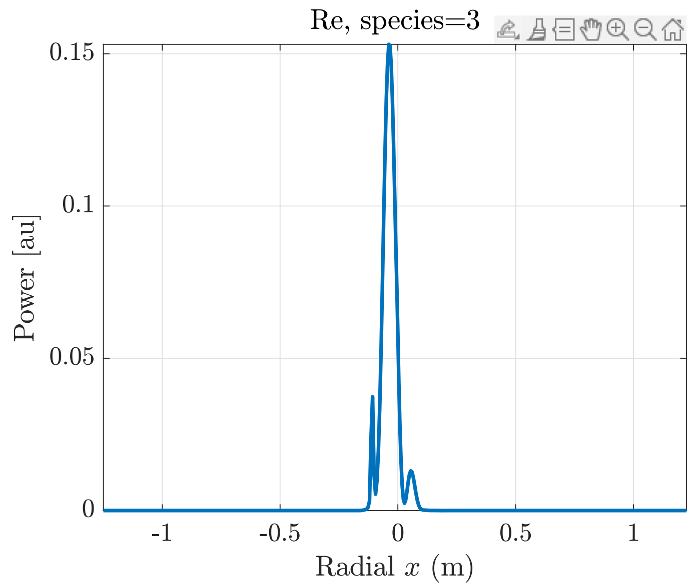
- Radial power absorption profiles



e$^-$ (Landau)



D$^+$ (numerical noise)



H$^+$ minority (cyclotron)

[Reman *et al*, RFPPC2025]

Main code developments in 2025

- PLIKES:

  - Continuous development effort and benchmarking,

  - Specific integration scheme for log-singular cyclotron kernel,

  - Parallelism, optimization towards HPC under way.

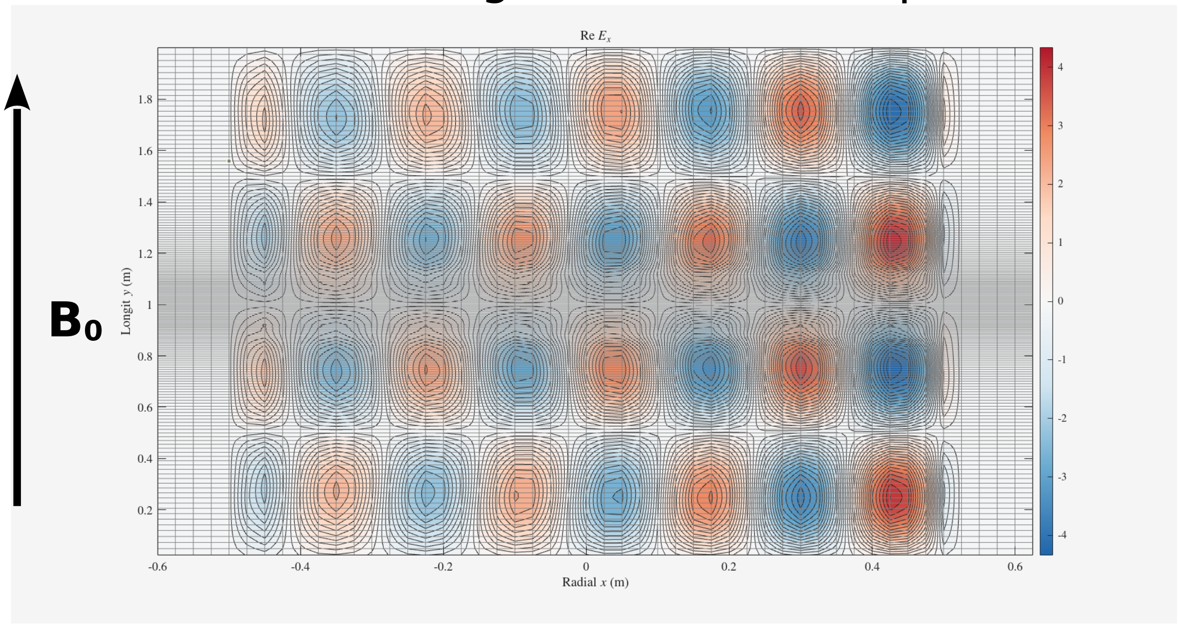
  - Capability for local mesh refinements along and across $\mathbf{B}_0$ implemented and tested on structured quad mesh.

  - Successful implementation of semi-analytical evaluation for the FE nonlocal plasma contributions

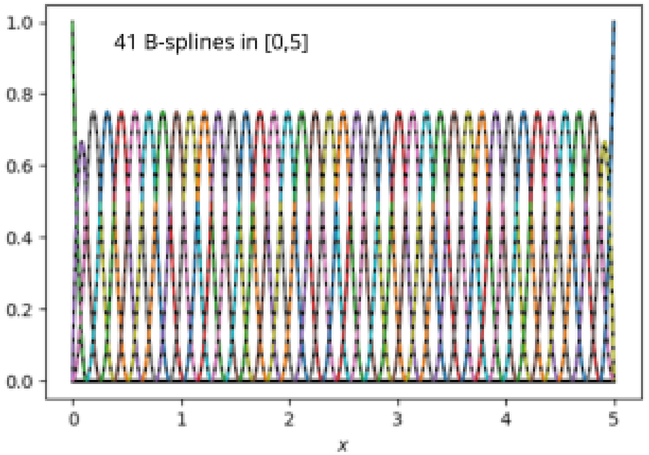
- Semi-analytical evaluation for the B-spline nonlocal plasma contributions ready for implementation into Psydac

- Steady progress on first FE implementation in the general Gmsh-FEM environment at ULiège

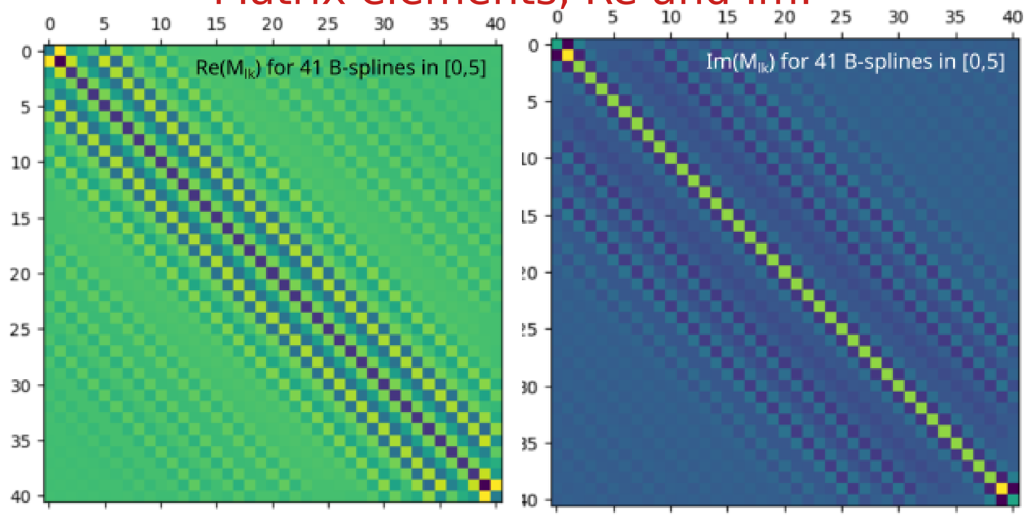
- PLIKES: local mesh refinement along $\mathbf{B_0}$ (structured quad mesh):



- Semi-analytical evaluation of B-spline nonlocal plasma contributions ready for implementation into Psydac:



Matrix elements, Re and Im:



ENR publications

- 25th Topical Conference on RF Power in Plasmas (Germany, May 2025):
  - "Integral dielectric kernel approach to modelling RF heating in toroidal plasmas", P. Lamalle *et al*, invited talk
  - "Integral dielectric kernel implementation to model RF heating in toroidal plasmas", B. Reman *et al*.

- Several internal presentations at LPP and IPP meetings

- Large theory manuscript (detailing and completing pre-ENR tokamak research), "High-frequency dielectric integral kernels for Maxwellian tokamak plasmas", soon to be submitted

- Manuscripts in preparation, reporting on
  - comprehensive properties of our kernel dispersion functions (2025),
  - first numerical demonstration of our approach with PLIKES (2026)
  - extension of the theory to stellarators,
  - new 'super-kernels' exploiting stellarator and tokamak symmetries
  - specific behaviour of the RF dielectric response at rational flux surfaces

Status of the ENR deliverables

- We have demonstrated the capability of our configuration space integral approach to treat warm plasma wave dispersion effects along $\mathbf{B_0}$, in simplified plasma slab geometry.
- Unexpected wealth of theory results obtained in 2025 proves highly beneficial.
- Final report will cover all aspects of the Project - still a month of work to go!
- PLIKES code reference version and documentation to be updated on gitlab.
- Besides conference presentations, publications are in preparation (delayed by intense ‘production phase’)
- The workplan was extremely ambitious for a 2-year period. We are most grateful to EUROfusion to be granted funding to pursue this line of research through 2026-2027.

  Primary goals: code extension to realistic toroidal geometries and exploitation.

Forthcoming developments (2026-2027): a graded approach

Top priority - toward realistic toroidal geometries:

- Exploit toroidal symmetries *à la* Jaeger 2002: 5 simulations on a single W7-X sector (using generalized periodicity conditions) $\Rightarrow$ RF field over whole device.
- In-house PLIKES code: field-aligned mesh in high favour; covariant mapping from FE to physical space will generalize 3D slab model.
- More general approach may suit Gmsh-FEM and Psydac.

- Code optimization ‘by all possible means’; scaling with problem size being documented (Pitagora).
- Essential to develop local mesh refinement / auto-adaptive meshing in view of future FLR.
- Scalability to very large problems: dedicated development in Gmsh-FEM.
- Our three FEM codes may call for different optimized solutions.

Preparing the next steps: mapping to realistic toroidal geometries

- We are so far using H(curl)-conforming vector finite elements in slab geometry
- Upgrade to general curvilinear geometry: covariant mapping from Cartesian reference finite element space $(\boldsymbol{r})$ to curved physical domain $(\boldsymbol{u})$,
  - Jacobian matrix $$J = \{J_{ij}\} = \left\{ \frac{\partial r_i}{\partial u^j} \right\}$$
  - Physical RF electric field $\boldsymbol{E}(\boldsymbol{r})$, components $\{E_i\}$
  - The covariant components $\{e_i\} = J^T \ \{E_i\}$ are discretized in the ref. FE space
  - The weak form is evaluated in the ref. FE space
  - Standard transformation of the curl operator:

$$\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}) = \frac{1}{\det J} \begin{vmatrix} \partial \boldsymbol{r}/\partial u^1 & \partial \boldsymbol{r}/\partial u^2 & \partial \boldsymbol{r}/\partial u^3 \\ \partial/\partial u^1 & \partial/\partial u^2 & \partial/\partial u^3 \\ e_1 & e_2 & e_3 \end{vmatrix}$$

- Preserves tangential component of basis functions on edges and facets
- We plan using a locally orthogonal magnetic-field-aligned coordinate system
- Other options being considered in the Psydac & Gmsh-fem / GetDP implementations

Stratton 1941, Šolín et al 2004, Scroggs (DefElement, online)

Forthcoming developments (2026-2027): a graded approach

Priorities for additional physics:

(theory: throughout 2026-2027; numerics: to be considered from mid-2027)

- FLR effects:

  - The above theory is available with FLR ‘full-wave’ operator expansion

  - Different possible approaches:

  ◦ Integral approach // & truncated expansion in powers of $\nabla_\perp, \nabla_\pm$: modifies partial differential operator, needs suitable FE basis functions.

  ◦ Integral approach // & $\perp$ integral operator similar to Machielsen’s in general geometry: $\perp$ nonlocality on the thermal LR scale.

- Non-Maxwellian RF response, with consistent QLFP diffusion coefficient.

Conclusions

- Theory & graded implementation with three concurrent FEM tools under vigorous development.
- Offers a configuration space integral approach to modelling // kinetic effects in toroidal devices, i.e. in presence of poloidal field and $\nabla_{//} B_0$.
- Derives from spectral theory & shares its physics contents, providing complementary viewpoint and specific advantages.
- Integral kernels obtained for Maxwellian tokamak & stellarator plasmas, properties investigated in detail,
- The ENR has allowed us to implement and validate the most essential features of our approach. Its extension to realistic toroidal geometries will follow.
- Staged development:

  - 2024-2025: proof-of-principle PLIKES code developed, basic numerical aspects dealt with, simplification with semi-analytical plasma contributions to system matrix;
  - 2026-2027: realistic stellarator (& tokamak), code optimization, meshing strategies;
  - Then, in a position to extend to FLR & other distribution functions;
  - ...

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- P. Popovich et al 2006, CPC **175** 250–263, “A full-wave solver of the Maxwell’s equations in 3D cold plasmas”
- N. Mellet et al 2011, CPC **182** 570–589, “Convolution and iterative methods applied to low-frequency waves in 3D warm configurations”

SCENIC applications:
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- C. Slaby et al 2025, Plasma Phys. Control. Fusion 67 065041C, “Schemes for generating deeply trapped fast ions via ion-cyclotron-resonance heating in Wendelstein 7-X”

Theory:
- P. Lamalle 2019, AIP Conference Proc. **2254** 1, 100001
- M. Machielsen et al 2023, Fundamental Plasma Physics **3** 100008
- P. Lamalle et al 2025, RFPPC2025 talk + to be submitted
- PLIKES code: B. Reman et al, RFPPC2025

PSYDAC:
- Y. Güçlü et al 2022, 8th ECCOMAS
- E. Moral Sanchez et al 2025, arXiv-2501
- Gmsh-FEM: A. Royer et al 2021, ECCOMAS, DOI: 10.23967/wccm-eccomas.2020.161

Acknowledgments

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