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Integral kernel approach to modelling wave heating of stellarator plasmas: breaking further ground in theory and numerical implementation

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The project

- "Integral kernel approach to modelling wave heating of stellarator plasmas: breaking further ground in theory and numerical implementation"
- Funded by EUROfusion theory & modelling ENR over 2026 - 2027 "CfP-FSD-AWP26-ENR-05-LPP-ERM-KMS-02"
- Continuation of our 2024-2025 ENR, with MPG Garching NMPP now officially included.

Background

- Waves in hot fusion plasmas: long range wave - particle interactions along equilibrium $\boldsymbol{B_0}$
- Kinetic description of hot plasma HF response in realistic toroidal geometries is challenging: rotational transform, curved $\boldsymbol{B_0}$, $\nabla_{//} B_0 \neq 0$.
- Very different world from $\perp$ - stratified, straight-$\boldsymbol{B_0}$ equilibria

  ⇒ Traditional approach to realistic full-wave modelling of wave heating in tokamaks and stellarators relies on Fourier expansions of the HF fields along 2 or 3 spatial coordinates.
- Indeed allows convenient kinetic theoretical treatment of wave dispersion along curved $\boldsymbol{B_0}$;

⇒
- Mixed spectral-finite element or fully spectral numerical formulations,
- Plasma described by dielectric tensor formulated in Fourier space involving the well-known $Z^{\{\alpha\}}$ dispersion functions for Maxwellians,
- Used in most ICRH codes: TORIC, CYRANO, EVE, AORSA, LEMan…

Reference set of equations: linearized Vlasov - Maxwell system

- Maxwell-Vlasov system (frequency domain):

$$\nabla \times \nabla \times \boldsymbol{E} = \left(\frac{\omega}{c}\right)^2 \left(\boldsymbol{I} + \frac{\mathrm{i}}{\omega\,\epsilon_0} \sum_\beta \boldsymbol{\sigma_\beta}\right) . \boldsymbol{E}$$

($\boldsymbol{E}$: RF electric field)

- RF current density of each species β:

$$\boldsymbol{j_\beta(r)} \equiv \boldsymbol{\sigma_\beta} \cdot \boldsymbol{E} = q_\beta \int f_\beta \boldsymbol{v}\, \mathrm{d}v^3$$

- Vlasov HF perturbed distribution function:

$$f_\beta(\boldsymbol{r},\boldsymbol{v}) = -\frac{q_\beta}{m_\beta} \int_{-\infty}^{t} \mathrm{e}^{-\mathrm{i}\omega(t'-t)} [\boldsymbol{E}(\boldsymbol{r}') + \boldsymbol{v}' \times \boldsymbol{B}(\boldsymbol{r}')] \cdot \frac{\partial f_{0\beta}}{\partial \boldsymbol{v}'}\, \mathrm{d}t'$$

Reference set of equations: linearized Vlasov - Maxwell system

- Maxwell-Vlasov system (frequency domain), 'weak' form:

$$\frac{\mathrm{i}}{2} \int_{\mathcal{V}} \left[\frac{1}{\omega\,\mu_0} (\nabla \times \boldsymbol{F})^* . (\nabla \times \boldsymbol{E}) - \omega\,\varepsilon_0\, \boldsymbol{F}^* . \boldsymbol{E} \right] \mathrm{d}r^3 + \sum_\beta \mathcal{W}_{\boldsymbol{FE}\beta} = -\frac{1}{2} \int_{\mathcal{V}} \boldsymbol{F}^* . \boldsymbol{j}_S\, \mathrm{d}r^3$$

($\boldsymbol{E}$: RF electric field, $\boldsymbol{F}$: arbitrary test function field, $\boldsymbol{F} \equiv \boldsymbol{E}$ ⇒ Poynting's theorem)

- This formulation emphasizes the dielectric response of each species β:

$$\mathcal{W}_{\boldsymbol{FE}\beta} = \frac{1}{2} \int_{\mathcal{V}} \boldsymbol{F}^* \cdot \boldsymbol{j}_\beta\, \mathrm{d}r^3 = \frac{q_\beta}{2} \int_{\mathcal{V}} \mathrm{d}r^3 \int \mathrm{d}v^3 f_\beta \boldsymbol{F}^* \cdot \boldsymbol{v}.$$

… rather than the RF current density $\boldsymbol{j_\beta(r)} \equiv \boldsymbol{\sigma_\beta} \cdot \boldsymbol{E} = q_\beta \int f_\beta \boldsymbol{v}\, \mathrm{d}v^3$

- Vlasov HF perturbed distribution function:

$$f_\beta(\boldsymbol{r},\boldsymbol{v}) = -\frac{q_\beta}{m_\beta} \int_{-\infty}^{t} \mathrm{e}^{-\mathrm{i}\omega(t'-t)} [\boldsymbol{E}(\boldsymbol{r}') + \boldsymbol{v}' \times \boldsymbol{B}(\boldsymbol{r}')] \cdot \frac{\partial f_{0\beta}}{\partial \boldsymbol{v}'}\, \mathrm{d}t'$$

- Interests:
  Theory: facilitates consistent treatment of geometry
  Applications: ideally suited for implementation in FEM codes & extraction of power balance

Notations

- Field components: left (+) and right (-) circular polarizations, parallel (//)

$$\boldsymbol{E} = E_+ \boldsymbol{e_+} + E_- \boldsymbol{e_-} + E_{//} \boldsymbol{e_{//}} = \sum_{L=-1}^{+1} E_{\mathcal{L}} \boldsymbol{e_{\mathcal{L}}}\,, \qquad \boldsymbol{F} = \sum_{L'=-1}^{+1} F_{\mathcal{L}'} \boldsymbol{e_{\mathcal{L}'}}$$

$\mathcal{L}$	+	−	//
L	+1	−1	0

- ' $v_{//}^{\alpha}$ index': $\alpha = \delta_{L,0} + \delta_{L',0}$ — $\alpha = 0$: cyclotron and TTMP; $\alpha = 2$: Landau; $\alpha = 1$: mixed Landau-TTMP
- Lowest order FLR: only 'diagonal' contributions $L = L'$, $\alpha = 2\delta_{L,0}$

  i.e. the bilinear dielectric response only involves $F_{\mathcal{L}}\, E_{\mathcal{L}}$ terms
- We use p for the cyclotron harmonic index:

  $p = +1$: ion fundamental, $p = 0$: Landau, $p = -1$: electron fundamental

Integral kernels in configuration space

- **Focus of our ENR**: alternative approach, with the plasma dielectric response formulated as a nonlocal integral operator in physical space, involving new Maxwellian '*kernel dispersion functions (KDF)*'.
- Max. analytical developments
  ⇒ extra physics insight and faster numerical simulations.
- Current emphasis on long-range dispersion effects along $\boldsymbol{B_0}$, outstanding in applications. Implementation in progress.
- N.B. For the sake of clarity, presentation only shows lowest order FLR.

Advantages of the configuration space integral approach

- Leaves complete freedom of choice for numerical discretization.

  ⇒ Enables FEM methods in 2D and 3D to model wave propagation and absorption in hot inhomogeneous fusion plasmas;
- Enables local mesh refinements (ruled out with spectral methods), essential to address FLR effects in 2D / 3D;
- Better suited field representations to deal with FLR in toroidal geometry;
- Straightforward connection with RF antenna models based on the FEM.

⇒ Main goals of the project:
- Efficient implementation in new full-wave code & existing FEM packages;
- Validation, demonstration of attractiveness, model RF heating in tokamaks and stellarators.

Earlier work on the configuration space approach

- Sauter & Vaclavik 1992, 1994, Smithe *et al* 1997: ⊥-stratified plasma, focus on ⊥ nonlocal effects, spectral in // direction
- Meneghini, Shiraiwa & Parker 2009: LHCD, integral treatment of Landau damping, iterative solution
- Svidzinski 2016: very general approach, hot conductivity kernel evaluated numerically by orbit integration
- Fukuyama, 2019 RFPPC
- Lamalle, 2019 RFPPC, 2023 EFTC, 2024 Varenna: tokamak theory, // treatment
- Machielsen, Rubin & Graves 2023: full FLR theory for homogenous plasmas, both // and ⊥ treatments. Applied to ⊥ but not yet to //.

Other recent treatments of // dispersion, using iterative methods: Vallejos *et al* 2018, Zaar *et al* 2024

Configuration space approach: infinite //-homogeneous plasmas

- Plasma response, showing 0th order FLR: involves nonlocal integrals (z, z') along magnetic field lines

$$\mathcal{W}_{\boldsymbol{FE}\beta} = -\frac{\mathrm{i}\varepsilon_0}{2} \sum_{L=-1}^{1} \delta_{L,p}\, 2^{\alpha/2} \int \mathrm{d}r_\perp^2\, \frac{\omega_\mathrm{p}^2}{v_\mathrm{T}} \int \mathrm{d}z \int \mathrm{d}z'\, F_{\mathcal{L}}^*(z')\, \Upsilon_\alpha(\lambda\,|z'-z|)\, E_{\mathcal{L}}(z) \qquad (\alpha = 2\delta_{L,0})$$

$$\lambda = \frac{\omega - p\,\omega_\mathrm{c}}{2\,v_\mathrm{T}}$$

The kernel dispersion functions (KDF) are derived from the usual PDF:

$$\Upsilon_\alpha(\xi) = \frac{1}{\pi} \int_0^{+\infty} Z^{\{\alpha\}}\left(\frac{2\,\xi}{t}\right) \begin{pmatrix} \cos t \\ \mathrm{i}\sin t \end{pmatrix} \frac{\mathrm{d}t}{t}, \qquad \alpha \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}, \quad \mathrm{Im}\,\xi > 0$$

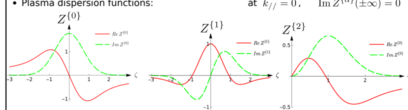
- $Z^{\{\alpha\}}$: standard plasma dispersion functions (PDF)
- Odd α kept for completeness, $\alpha = 1$ enters the FLR theory
- The form of Υ_0 appears in Svidzinski (2016)'s conductivity kernel.

  Lamalle (2023, 2024), equivalent to Machielsen *et al* (2023) S_1, S_2, S_3 but ≠ def.

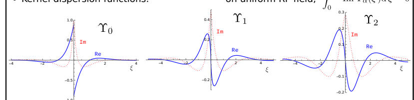
[Lamalle *et al*, RFPPC2025]

Correspondence

- Plasma dispersion functions: at $k_{//} = 0$, $\mathrm{Im}\, Z^{\{\alpha\}}(\pm\infty) = 0$



- Kernel dispersion functions: on uniform RF field, $\int_0^\infty \mathrm{Im}\,\Upsilon_\alpha(\xi)\,\mathrm{d}\xi = 0$



Cyclotron (p≠0), TTMP
Im log singular at $\xi = 0$

Mixed TTMP-Landau

Landau (p=0)

Detailed analytical study of the KDF

Latest analytical results (2025):

- The following double // integrals are evaluated analytically in terms of the same family of special functions:

$$(z_2 - z_1)(z_2' - z_1')\, D_\alpha^{(m,n)} = \int_{z_1'}^{z_2'} \mathrm{d}z' \left(\frac{z' - z_1'}{z_2' - z_1'}\right)^m \int_{z_1}^{z_2} \mathrm{d}z\, \Upsilon_\alpha(\beta\,|z - z'|) \left(\frac{z - z_1}{z_2 - z_1}\right)^n$$

- A KDF of arbitrary index can be evaluated in terms of $\Upsilon_0, \Upsilon_1, \Upsilon_2$ alone.
- The nonlocal interactions between FE (or B-spline) basis functions can be evaluated semi-analytically, using the $D_\alpha^{(m,n)}$ as building blocks.

  ⇒ Code simplification and strong acceleration!
- Detailed theory paper on the KDF soon to be submitted for publication.

Tokamak: full configuration space result

- Analytical developments remove the poloidal & toroidal Fourier expansions.
- Plasma response result, showing 0th order FLR: involves nonlocal integrals (*s, s'*) along magnetic field lines,

$$\mathcal{W}_{\boldsymbol{FE}\beta} = -\frac{\mathrm{i}\varepsilon_0}{2} \sum_{L=-1}^{1} \delta_{L,p}\, 2^{\alpha/2} \int \mathrm{d}r_\perp^2\, \frac{\omega_\mathrm{p}^2}{v_\mathrm{T}} \int \mathrm{d}s \int \mathrm{d}s'\, F_{\mathcal{L}}^*(s')\, \Upsilon_\alpha\left[\hat{\xi}_p(s,s')\right] E_{\mathcal{L}}(s) \qquad (\alpha = 2\delta_{L,0})$$

- These are the same KDFs as for infinite homogeneous plasmas!
- Here, their argument is evaluated at the mid-point between field (*s*) and test (*s'*) points:

$$\hat{\xi}_p(s,s') = \frac{\omega - p\omega_\mathrm{c}(\frac{s+s'}{2})}{v_\mathrm{T}}\, \frac{|s'-s|}{2}$$

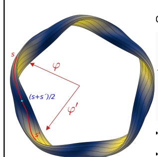
- This is essentially a triple integral over plasma volume + an extra integral accounting for // nonlocality. (NB with suitable Jacobians in curved geometries)

Stellarator: configuration space result

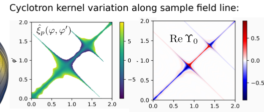
- Same method applied to spectral Maxwellian plasma response of [Vdovin 1996, Fukuyama 2000, Murakami 2006]* ⇒ same formal expression as for tokamaks:

$$\mathcal{W}_{\boldsymbol{FE}\beta} = -\frac{\mathrm{i}\varepsilon_0}{2} \sum_{L=-1}^{1} \delta_{L,p}\, 2^{\alpha/2} \int \mathrm{d}r_\perp^2\, \frac{\omega_\mathrm{p}^2}{v_\mathrm{T}} \int \mathrm{d}s \int \mathrm{d}s'\, F_{\mathcal{L}}^*(s')\, \Upsilon_\alpha\left[\hat{\xi}_p(s,s')\right] E_{\mathcal{L}}(s) \qquad (\alpha = 2\delta_{L,0})$$

$$\hat{\xi}_p(s,s') = \frac{\omega - p\omega_\mathrm{c}(\frac{s+s'}{2})}{v_\mathrm{T}}\, \frac{|s'-s|}{2}$$



Cyclotron kernel variation along sample field line:



- Interaction mostly involves close neighbour points, except near 2 cyclotron layers
- Helps determine system sparsity pattern for 3D stellarator geometry

* Ignoring drift waves and specific $\partial f_0/\partial\psi$ effects; and assuming integrable orbits.

Progressive FEM implementation

- Following three paths:
  - In-house PLIKES code, 2.5D slab model (quadratic Nédélec+Lagrange)
  - NMPP Garching's Psydac (tensor product B-splines): implementation under way.
  - ULiège's Gmsh-FEM (high degree polynomials): in progress, PhD started in October 2024, Specific goal: enabling / optimizing very large scale computing. Linear system preconditioning, domain decomposition and iterative methods (innovative for Maxwell's equations).
- Staged development, initially FLR^0 ⇒ minority & 3-ion ICRH scenarios

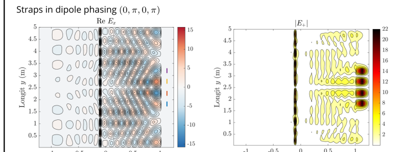
[Reman *et al*, RFPPC2025]

Sample PLIKES ICRH simulation

⊥-stratified plasma (2-ion hybrid layer at x=-0.1m)

Integral dielectric kernel to JET(like) H minority heating in D.

Straps in dipole phasing $(0, \pi, 0, \pi)$



RF field radial component — Left-hand circular RF field polarization

The antenna toroidal spectrum is captured in a single finite element simulation.

Status

- We have so far demonstrated the capability of our configuration space integral approach to treat warm plasma wave dispersion effects along $\mathbf{B_0}$, in simplified plasma slab geometry.
- Theory & graded implementation with three concurrent FEM tools under vigorous development.
- Offers a configuration space integral approach to modelling // kinetic effects in toroidal devices, i.e. in presence of poloidal field and $\nabla_{//} B_0$.
- Derives from spectral theory & shares its physics contents, providing complementary viewpoint and specific advantages.
- Integral kernels obtained for Maxwellian tokamak & stellarator plasmas, properties investigated in detail, unexpected analytical results obtained in 2025 highly beneficial.
- Extension to realistic toroidal geometries under way in 2026-2027.

Forthcoming developments (2026-2027): a graded approach

- Primary goals: code extension to realistic toroidal geometries and exploitation.
- Exploit toroidal symmetries *à la* Jaeger 2002: e.g. 5 simulations on a single W7-X sector (using generalized periodicity conditions) ⇒ RF field over whole device.
- In-house PLIKES code: field-aligned mesh in high favour; covariant mapping from FE to physical space will generalize 3D slab model.
- Other approaches may suit Gmsh-FEM and Psydac.
- Code optimization 'by all possible means'; scaling with problem size being documented (PLIKES on Pitagora).
- Essential to develop local mesh refinement / auto-adaptive meshing in view of future FLR physics.
- Scalability to very large problems: dedicated developments in Gmsh-FEM.
- The three FEM codes may call for different optimized solutions.

Forthcoming developments (2026-2027): a graded approach

Priorities for additional physics, on the longer term:
(theory: throughout 2026-2027; numerics: to be considered from mid-2027)

- FLR effects:
  - The theory is available with FLR 'full-wave' operator expansion
  - Different possible approaches:
    - Integral approach // & truncated expansion in powers of $\nabla_\perp, \nabla_\pm$: modifies partial differential operator, needs suitable FE basis functions.
    - Van Eester & Budé - like approach, see e.g. [Van Eester et al, Maquet et al, RFPPC2025]: integro-differential operator obtained from polynomial fit in ***k*** space.
    - Integral approach // & ⊥ integral operator similar to Machielsen's in general geometry: ⊥ nonlocality on the thermal LR scale.
  - First step: carefully balance pros and cons, and select one.
- Non-Maxwellian RF response, with consistent QLFP diffusion coefficient.

QR codes to our contributions at the 2025 RF Power in Plasmas Conference (Hohenkammer, Germany)

P. Lamalle *et al*, invited presentation:

B. Reman *et al*, proceedings paper and poster: