1. Excitation of TAE modes using an antenna in ORB5
2. Convolution-based solution of Ito process

Mohsen Sadr

Collaborators: L. Villard, S. Brunner, P. Donnel, M. Murugappan, A. Biancalani, A. Bottino, T. Hayward-Schneider, A. Mishchenko.

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ORB5

Consider linear Gyro-kinetic particle description

$$\dot{\boldsymbol{X}} = \boldsymbol{v}_{||} + \boldsymbol{v}_c + \boldsymbol{v}_{\nabla B}, \tag{1.1}$$

$$\dot{v}_z = -\frac{\boldsymbol{v}_{||} + \boldsymbol{v}_c}{m_\sigma v_z} \cdot \nabla(\mu B) \tag{1.2}$$

where

$$\boldsymbol{v}_{||} := v_z \boldsymbol{b}, \tag{1.3}$$

$$\boldsymbol{v}_c := \frac{m_\sigma v_z^2}{q_\sigma B_{||}^*} \boldsymbol{b} \times (\boldsymbol{b} \cdot \nabla \boldsymbol{b}), \tag{1.4}$$

$$\boldsymbol{v}_{\nabla B} := \frac{\mu}{q_\sigma B_{||}^*} \boldsymbol{b} \times \nabla B. \tag{1.5}$$

Here, $\boldsymbol{B}$ is a pre-described and unperturbed ad-hoc magnetic field with poloidal flux function $\psi = \psi(r)$ and $d\psi/dr = rB_0/\overline{q}$.

Excitation of TAE modes using antenna in ORB5

Let us introduce an antenna to the system

$$H = H_0 + \mathcal{O}(\epsilon_\delta) + \epsilon_{\mathrm{ant}} H_{1,\mathrm{ant}} + \mathcal{O}(\epsilon_{\mathrm{ant}}^2) \tag{1.6}$$

where

$$H_{1,\mathrm{ant}} = q_\sigma \langle \phi_{\mathrm{ant}} - v_z A_{1||\mathrm{ant}} \rangle \tag{1.7}$$

Taking variational derivatives of ORB5 action with this modified Hamiltonian leads to gyrokinetic Vlasov eq. with characteristics

$$\dot{\boldsymbol{X}} = \boldsymbol{v}_{||} + \boldsymbol{v}_c + \boldsymbol{v}_{\nabla B} + \epsilon_{\mathrm{ant}} \boldsymbol{v}_{E_{\mathrm{ant}} \times B} \cancel{+ \epsilon_{\mathrm{ant}} \boldsymbol{v}_{A_{\mathrm{ant}}}}, \tag{1.8}$$

$$\dot{v}_z = -\frac{(\boldsymbol{v}_{||} + \boldsymbol{v}_c)}{m_\sigma v_z} \cdot \nabla \left(\mu B + \epsilon_{\mathrm{ant}} q_\sigma \langle \phi_{\mathrm{ant}} \cancel{- v_z A_{1||\mathrm{ant}}} \rangle \right). \tag{1.9}$$

Here we consider

$$\phi_{\mathrm{ant}} = s e^{-(\frac{s-s_0}{\delta})^2} \mathrm{Re} \left[\sum_j^N A_j e^{\hat{i}(m_j\theta + n_j\phi)} e^{\hat{i}\omega_{\mathrm{ant}} t} \right]. \tag{1.10}$$

$s_0 = 0.5\ \delta = 0.1\ A_j = 10^{-6}\ n_j = 6, 6\ m_j = -10, -11$

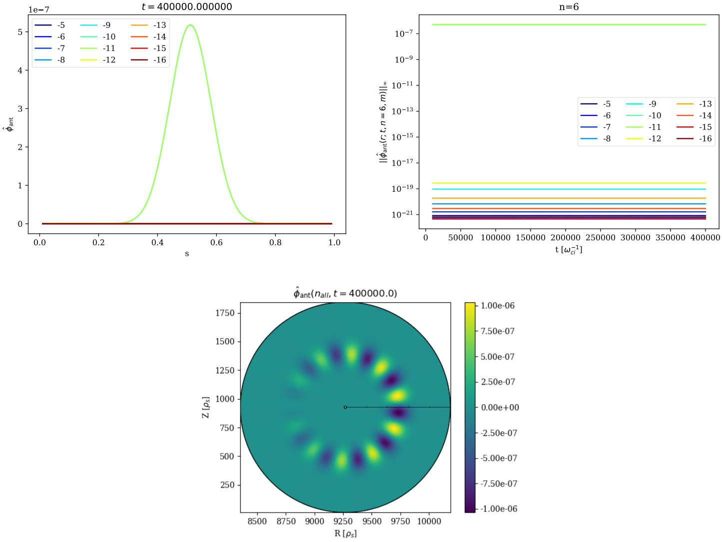
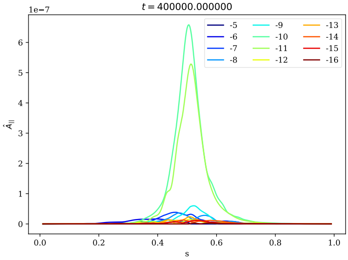
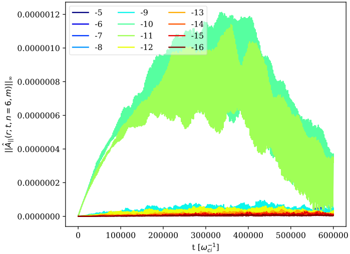


Figure: Antenna's potential profile

$s_0 = 0.5\ \delta = 0.1\ A_j = 10^{-6}\ n_j = 6, 6\ m_j = -10, -11$





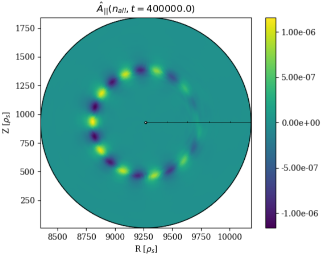
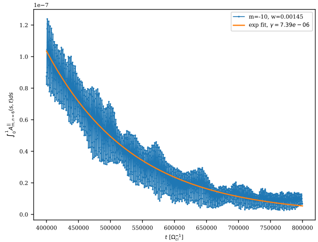
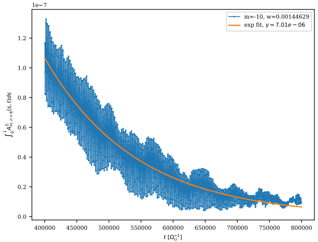
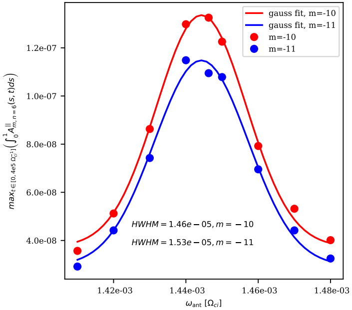


Figure: $A^{||}$

$s_0 = 0.5$ $\delta = 0.1$ $A_j = 10^{-6}$ $n_j = 6, 6$ $m_j = -10, -11$







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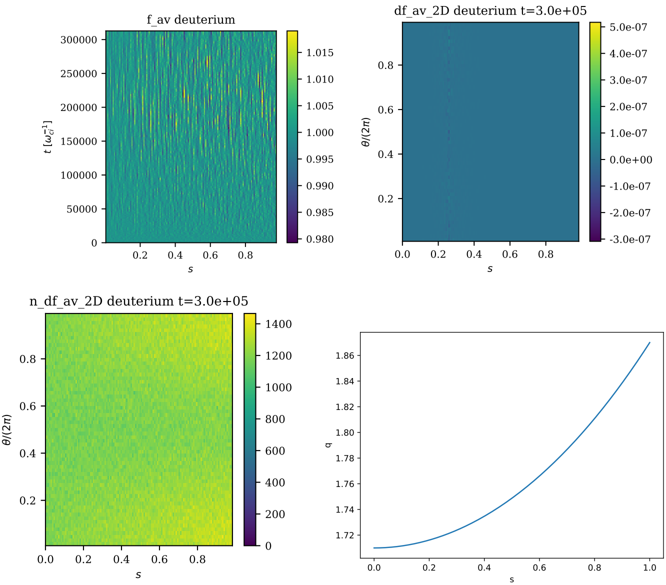

Next:

- Interaction of fast particles with antenna.
- Steeper safety factor profile.
- EM simulations.
- Implementing antenna as $A^{||}$ in ORB5.

2. Convolution-based solution of Ito process

Consider Ito process of $\boldsymbol{V}$ with drift $\boldsymbol{A}$ and diffusion $\boldsymbol{D} = \frac{1}{2}\boldsymbol{\sigma}\boldsymbol{\sigma}^T$, i.e.

$$dV_i = A_i dt + \sigma_{ij} dW_j \tag{2.1}$$

$$\text{Euler} - \text{Maruyama}: \ \Delta V_i = A_i^{(n)} \Delta t + \sigma_{ij}^{(n)} \mathcal{R}_j(0, \Delta t) \tag{2.2}$$

$$= \mathcal{R}_j \left(A_i^{(n)} \Delta t, D_{ij}^{(n)} \Delta t \right) \tag{2.3}$$

$$\implies V_i^{(n+1)} = V_i^{(n)} + \mathcal{R}_j \left(A_i^{(n)} \Delta t, D_{ij}^{(n)} \Delta t \right) \tag{2.4}$$

We know pdf of $X + Y$ is $f_{X+Y}(z) = \int f_X(\xi) f_Y(z - \xi) d\xi$ if X and Y are ind. rand. var.

$$f_{\boldsymbol{V}^{(n+1)}}(\boldsymbol{\eta}) = \int f_{\boldsymbol{V}^{(n)}}(\boldsymbol{\xi}) \mathcal{N} \left(\boldsymbol{\eta} - \boldsymbol{\xi} | \boldsymbol{A}^{(n)}(\boldsymbol{\xi}) \Delta t, \boldsymbol{D}^{(n)}(\boldsymbol{\xi}) \Delta t \right) d^n \boldsymbol{\xi}. \tag{2.5}$$

The multivariate Gaussian ditr.

$$\mathcal{N}\left(\boldsymbol{\eta}-\boldsymbol{\xi}|\boldsymbol{a},\boldsymbol{\Sigma}\right)=\frac{e^{-\frac{1}{2}(\boldsymbol{\eta}-\boldsymbol{\xi}-\boldsymbol{a})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\eta}-\boldsymbol{\xi}-\boldsymbol{a})}}{(2\pi)^{n/2}\sqrt{\det(\boldsymbol{\Sigma})}},\quad \text{where} \tag{2.6}$$

$$\boldsymbol{a}(\boldsymbol{\xi})=\Delta t\boldsymbol{A}(\boldsymbol{\xi})\ \text{ and }\ \boldsymbol{\Sigma}(\boldsymbol{\xi})=\Delta t\boldsymbol{D}(\boldsymbol{\xi}), \tag{2.7}$$

using eigenvalue decomposition $\boldsymbol{\Sigma}^{-1}=\boldsymbol{U}^T\boldsymbol{\Lambda}\boldsymbol{U}$ can be written

$$\mathcal{N}\left(\boldsymbol{\eta}-\boldsymbol{\xi}|\boldsymbol{a},\boldsymbol{\Sigma}\right)=\frac{e^{-(\boldsymbol{\eta}-\boldsymbol{\xi}-\boldsymbol{a})^T\boldsymbol{U}^T\boldsymbol{\Lambda}\boldsymbol{U}(\boldsymbol{\eta}-\boldsymbol{\xi}-\boldsymbol{a})/2}}{(2\pi)^{n/2}\sqrt{\det(\boldsymbol{\Sigma})}} \tag{2.8}$$

$$=\frac{e^{-|\hat{\boldsymbol{\eta}}-\hat{\boldsymbol{\xi}}-\hat{\boldsymbol{a}}|^2}}{(2\pi)^{n/2}\sqrt{\det(\boldsymbol{\Sigma})}},\quad \text{where } \hat{x}_i:=\sqrt{\lambda_i/2}(\sum_j^n U_{ij}x_j).$$

The exponential can be expanded with Hermite polynomials

$$e^{-|\hat{\boldsymbol{\eta}}-\hat{\boldsymbol{\xi}}-\hat{\boldsymbol{a}}|^2}=\sum_{\alpha\geq 0}^{p}\frac{1}{\alpha!}\left(\hat{\boldsymbol{\xi}}+\hat{\boldsymbol{a}}\right)^{\alpha}h_{\alpha}\left(\hat{\boldsymbol{\eta}}\right), \tag{2.9}$$

$$\Longrightarrow\ f_{\boldsymbol{V}^{(n+1)}}(\boldsymbol{\eta})=\sum_{\alpha\geq 0}^{p}\frac{1}{\alpha!}h_{\alpha}\left(\hat{\boldsymbol{\eta}}\right)\int\frac{f_{\boldsymbol{V}^{(n)}}(\boldsymbol{\xi})}{(2\pi)^{n/2}\sqrt{\det(\boldsymbol{\Sigma})}}\left(\hat{\boldsymbol{\xi}}+\hat{\boldsymbol{a}}\right)^{\alpha}d^n\boldsymbol{\xi}.$$

Linear Landau operator